SOPHIS

UNIVERSITE PIERRE & MARIE CURIE

Pricing of Asian Option on Commodity Basket

David Herrenschmidt

Master of Probability and Finance
(Internship Report)

September 2006

Internship Manager : Gregory Cobena

Contents

Abstract. *Interest for commodities and more especially for commodity derivatives has dramatically improved these last years. One of the most successful product is the asian option on basket of commodities. However, although the components of the basket might use a well known log-normal distribution Model, there is no closed form solution for the pricing. The main drawback of Monte-Carlo simulations, its slowness, becomes critical as commodity baskets have a large number of underlyings. Several analytical approximations rose. In the first part of this paper, we present and compare some of these approximations. Then we will detail the implementation issues on a "real world" example, namely the optimized greeks of the pricing method implemented in the Sophis Software.*

1 Introduction

Interest for commodities and more especially for commodity derivatives has dramatically improved these last years. One of the most successful product is the asian option on basket of commodities.

First, it allows industrial exposed to variations of several commodities to hedge with a single financial product. For example, an aircarft manufacturing company can hedge with a single basket option: (i) Aluminium prices, which represents a risk for manufacturing costs, (ii) Oil prices, which impact the sales and (iii) EUR/USD change which impact both the revenues and expenses.

Second, the basket option is typically a cheaper hedge than the purchase of options for each component of the basket since the correlations are below 100%.

Finally, note that Asian options, which are options on average prices over a pre-defined time-period are the most commonly used options in the commodity markets. They protect against manipulation of prices and reduce the cost of options by decreasing the volatility of the underlying.

In the current paper, the components of the basket are commodity Futures but methods presented can be adapted to different kinds of underlyings. For example, the Sophis model described below can also be used to add forward currencies values and forward FX rates in the basket definition. Moreover, we noted that the results of the experiments, in particular the comparison of benefits/drawbacks of the different models can be extended to other kinds of basket components (Equities, ...).

In this paper, we use the Black [Bla76] model for commodity Futures. This well known log-normal distribution Model can be used to price vanilla options for each underlying. Unfortunately, the sum of lognormal distributions is not a lognormal distribution, and there is no known closed form solution for the price of the basket asian option. It is possible to use Monte-Carlo simulations, but the main drawback of this solution is its slowness. This issue becomes critical when considering the greeks analysis. First Monte-Carlo simulations are not the best choice for computing the greeks. And also, in commodity basket options, it is frequent to have several tens or hundreds of commodities or Future contract. For example the GSCI Index [Sac06] is an average over 40 different commodities and possibly two Futures for each commodity. Hence, the number of underlyings of the option is very large (tens to hundreds), and high performance algorithms should be preferred to compute the greeks.

In the first part of the paper, we present briefly different analytical approximations. Then, we compare the accuracy of their prices in different conditions. In the second part of the paper, we present some work conducted on a "real world" example, namely the pricing algorithm that has been implemented in Sophis Risque v5.0.5 for commodity basket asian options. We studied the "optimized-greeks" that were computed by the application and extended the delta formulas to include the volatility smile effects.

2 Pricing methods

In this section, we present the different pricing methods.

The first three methods that we present, Sophis [Sop06], Gentle [Gen93] and Levy [Lev92], propose lognormal distributions to approximate of the distribution the underlying and then compute the price of the option using Black Formula.

Ju [Ju02] add a correction term to Levy approximation using Taylor expansion of the characteristic function of the underlying around zero volatilities.

Curran [Cur94] separates the price of the option in to parts by conditioning. The first part can be computed exactly, the second part requires Levy approximation and numerical integration.

Implementation : Sophis implemented its own method in its software. The other methods that we present are the most common approximations (Gentle, Levy, Ju, Curran). We implemented all these methods in a limited framework. We also implemented a Monte-Carlo pricing. This is used to verify the correctness of our implementations and also to conduct the experiments that are described in section 3.

2.1 Environment

First of all, let's fix the framework of our study. We aims at pricing asian options on basket commodities. Underlyings of these options are Future contracts on commodities. We will choose the Black model to describe the dynamic of underlyings.

In this model, p future F_i evolve according to the following stochastic differential equation under the risk neutral probability :

$$dF_i(t) = \sigma_i F_i(t) dW_t^i \tag{1}$$

where σ_i is the volatility of F_i supposed constant and $\{W_t^i, t \geq 0\}$ is a multidimensional standard Brownian motion under the risk neutral probability. Correlation between $W_t^{i_1}$ and $W_t^{i_2}$ is noted $\rho_{i_1 i_2}$.

Solving this equation we obtain :

$$F_i(t) = F_i(0)exp(-\frac{1}{2}\sigma_i^2 t + \sigma_i W_t^i) \tag{2}$$

Let's now consider n + 1 time $0 = t_0 < t_1 < ... < t_j < ... < t_n = T$ and the discrete arithmetic average :

$$A(T) = \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{p} F_i(t_j) \tag{3}$$

The payoff of the option we will study from now on is :

$$V(T) = (A(T) - K)^+ \tag{4}$$

where T is the maturity and K is the strike of the option. The option considered is European.

The price of this option at $t_0 = 0$ is given by :

$$V(0) = e^{-rT} E\left[(A(T) - K)^+\right] \tag{5}$$

where r is the interest rate supposed constant.

We will also need to price this option within the averaging period at date t with $t_k \leq t < t_{k+1}$. Then the value of $F_i(t_j)$ for $j \leq k$ is fixed and we can write the payoff of the option this way :

$$(A(T) - K)^+ = \frac{n-k}{n}\left(\frac{1}{n-k}\sum_{j=k+1}^{n}\sum_{i=1}^{p} F_i(t_j) - \frac{n-k}{n}(K - \sum_{j=k+1}^{n}\sum_{i=1}^{p}\frac{1}{n}F_i(t_j))\right)^+ \tag{6}$$

So we again have to price a asian basket option before the beginning of the averaging period. It is an option with strike $\frac{n-k}{n}(K - \sum_{j=k+1}^{n}\sum_{i=1}^{p}\frac{1}{n}F_i(t_j))$ averaged on date $t_{k+1}, ..., t_n$.

From now on we will consider the price of the option at date 0 with no loose of exhaustivity.

2.2 sophis

The basic idea is to approximate the distribution of $A(T)$ by a lognormal distribution ($A(T) = Mexp(-\frac{1}{2}\sigma^2 T + \sigma W_T)$). Then we will be able to apply the Black formula for a call.

The approximate distribution is fully determined by it's expectation M and it's volatility σ.

The most natural candidate to estimate M seems to be the expectation of A(T) :

$$M = E\left[A(T)\right] = \sum_{i=1}^{p} F_i(0) = A(0) \tag{7}$$

For the estimation of σ the arithmetic average is approximated by the following geometric average :

$$G(T) = \prod_{j=1}^{n}\prod_{i=1}^{p}\left(\frac{1}{n\alpha_i}F_i(t_j)\right)^{\alpha_i} \tag{8}$$

where

$$\alpha_i = \frac{\frac{1}{n}F_i(0)}{\sum_{i=1}^{p} F_i(0)} \tag{9}$$

The value of σ we be set equal to the volatility of $G(T)$ which is easily computable.

$$G(T) = \frac{1}{n}\prod_{i=1}^{p}\prod_{j=1}^{n}\left(\frac{F_i(0)}{\alpha_i}\right)^{\alpha_i} exp(-\frac{1}{2}\alpha_i\sigma_i^2 t_j + \alpha_i\sigma_i W_{t_j}^i) \tag{10}$$

$$= A(0)exp\left(\sum_{j=1}^{n}\sum_{i=1}^{p}(-\frac{1}{2}\alpha_i\sigma_i^2 t_j + \alpha_i\sigma_i W_{t_j}^i)\right) \tag{11}$$

we finally get :

$$\sigma^2 T = \sum_{j_1,j_2=1}^{n}\sum_{i_1,i_2=1}^{p} \alpha_{i_1}\alpha_{i_2}\sigma_{i_1}\sigma_{i_2}\rho_{i_1 i_2} min(t_{j_1}, t_{j_2}) \tag{12}$$

Then, we can apply the black formula for a call :

$$V(0) = e^{-rT}\left(M\mathcal{N}(d_1) - K\mathcal{N}(d_2)\right) \tag{13}$$

where :

$$d_1 = \frac{log(M) - log(K)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T} \tag{14}$$

$$d_2 = d_1 - \sigma\sqrt{T} \tag{15}$$

2.3 Gentle

The basic idea of Gentle is to approximate the arithmetic average $A(T)$ by the geometric $G(T)$ introduced for the Sophis model in the previous section. This geometric average presents the advantage of following a lognormal distribution ($G(T) = Mexp(-\frac{1}{2}\sigma^2 T + \sigma W_T)$).

The volatility of $G(T)$ is computed in the previous section and is given by :

$$\sigma = \frac{1}{T}\sqrt{\sum_{j_1,j_2=1}^{n}\sum_{i_1,i_2=1}^{p}\alpha_{i_1}\alpha_{i_2}\sigma_{i_1}\sigma_{i_2}\rho_{i_1 i_2}min(t_{j_1},t_{j_2})} \tag{16}$$

The expectation of $G(T)$ is of course different from the one of $A(T)$. It is given by :

$$M = E\left[G(T)\right] = A(0)exp\left(\sum_{j=1}^{n}\sum_{i=1}^{p}(-\frac{1}{2}\alpha_i\sigma_i^2 t_j) + \sigma^2 T\right) \tag{17}$$

Then, Gentle make the following approximation for the price of the option in order to compensate the difference between the expectation of $A(T)$ and the expectation of $B(T)$:

$$\begin{aligned} V(0) = e^{-rT}E\left[(A(T)-K)^+\right] &= e^{-rT}E\left[(G(T)-(K-A(T)+G(T)))^+\right] \\ &\approx e^{-rT}E\left[(G(T)-(K-E[A(T)]+E[G(T)]))^+\right] \end{aligned} \tag{18}$$

The call on $A(T)$ of strike K is approximated by a call on $G(T)$ of strike $\tilde{K} = K - E[A(T)] + E[G(T)]$. We can then apply the Black formula for a call.

2.4 Levy

The basic idea of Levy is to approximate the distribution of $A(T)$ by a lognormal distribution. To determine the expectation M and the volatility σ of this distribution, he proposes a pure moment matching technique.

Let's consider that :

$$A(T) = Mexp(-\frac{1}{2}\sigma^2 T + \sigma W_T) \tag{19}$$

then :

$$M = E[A(T)] = A(0) \tag{20}$$

and

$$E\left[A^2(T)\right] = M^2 exp(\sigma^2 T) \tag{21}$$

$$\Rightarrow \sigma^2 T = log\left(E\left[A^2(T)\right]\right) - 2log(M) \tag{22}$$

where :

$$E\left[A^2(T)\right] = \frac{1}{n^2} \sum_{j_1,j_2=1}^{n} \sum_{i_1,i_2=1}^{p} F_{i_1}(0) F_{i_2}(0) exp(\sigma_{i_1}\sigma_{i_2}\rho_{i_1 i_2} min(t_{j_1}, t_{j_2})) \tag{23}$$

Then we can apply the Black formula for Call.

2.5 ju

Several articles shows that two moment matching analytical approximations (as Levy) give good results for short maturity and weak volatility but are more discussable for long maturity and high volatility. To have better approximation, Ju proposes to consider the Taylor expansion around zero volatility of the ratio of the characteristic function of $log(A(T))$ to the characteristic function of logarithm of the lognormal approximation proposed by Levy.

To respect the relative weights of volatility performing the Taylor expansion, let's consider :

$$F_i(t,z) = F_i(0) exp(-\frac{1}{2} z^2 \sigma_i^2 t + z\sigma_i W_t^i) \tag{24}$$

and :

$$A(T,z) = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{p} F_i(t_j, z) \tag{25}$$

Let's note :

$$M_1 = E[A(T,z)] = \sum_{i=1}^{p} F_i(0) \tag{26}$$

and

$$M_2(z^2) = E[A^2(T,z)] \tag{27}$$

If we consider a normal random variable $Y(z)$ with mean $m(z^2)$ and variance $v(z^2)$, matching the first two moments of $exp(Y(z))$ with those of $A(T,z)$ we obtain :

$$m(z^2) = 2log(M_1) - \frac{1}{2}log(M_2(z^2)) \tag{28}$$

and

$$v(z^2) = log(M_2(z^2)) - 2log(M_1) \tag{29}$$

Let's note $X(z) = log(A(T,z))$. It's characteristic function is given by :

$$E\left[e^{i\theta X(z)}\right] = E\left[e^{i\theta Y(z)}\right]\frac{E\left[e^{i\theta X(z)}\right]}{E\left[e^{i\theta Y(z)}\right]} = E\left[e^{i\theta Y(z)}\right]f(z) \tag{30}$$

Let's expand $f(z)$ until z^6. Noting that $v'(z^2) = -2m'(z^2)$ we have got :

$$\frac{1}{E\left[e^{i\theta Y(z)}\right]} = exp\left(-i\theta m(z^2) + \frac{1}{2}\theta^2 v(z^2)\right) \tag{31}$$

$$= exp\left(-i\theta m(0) + \frac{1}{2}\theta^2 v(0) - (i\theta+\theta^2)(m'(0)z^2 + m''(0)\frac{z^4}{2} + m^{(3)}(0)\frac{z^6}{6}) + o(z^6)\right)$$

$$= e^{-i\theta m(0)+\frac{1}{2}\theta^2 v(0)}\left(1 - \tilde{\theta}a_1(z) + \tilde{\theta}^2\frac{a_1^2(z)}{2} - \tilde{\theta}\frac{a_2(z)}{2} + \tilde{\theta}^2\frac{a_1(z)a_2(z)}{2} - \tilde{\theta}\frac{a_3(z)}{6} - \tilde{\theta}^3\frac{a_1^3(z)}{6}\right)$$

where

$$\tilde{\theta} = i\theta + \theta^2 \tag{32}$$

and

$$a_1(z) = z^2 m'(0) = -\frac{M_2'(0)}{2M_2(0)}z^2 \tag{33}$$

$$a_2(z) = z^4 m''(0) = 2a_1^2(z) - \frac{M"_2(0)}{2M_2(0)}z^4 \tag{34}$$

$$a_3(z) = z^6 m^{(3)}(0) = 6a_1(z)a_2(z) - 4a_1^3(z) - \frac{M_2^{(3)}(0)}{2M_2(0)}z^6 \tag{35}$$

and

$$M_2(0) = \frac{1}{n^2} \sum_{j_1,j_2=1}^{n} \sum_{i_1,i_2=1}^{p} F_{i_1}(0)F_{i_2}(0) \tag{36}$$

$$M_2'(0) = \frac{1}{n^2} \sum_{j_1,j_2=1}^{n} \sum_{i_1,i_2=1}^{p} F_{i_1}(0)F_{i_2}(0)\sigma_{i_1}\sigma_{i_2}\rho_{i_1 i_2} min(t_{j_1}, t_{j_2}) \tag{37}$$

$$M_2''(0) = \frac{1}{n^2} \sum_{j_1,j_2=1}^{n} \sum_{i_1,i_2=1}^{p} F_{i_1}(0)F_{i_2}(0) \left(\sigma_{i_1}\sigma_{i_2}\rho_{i_1 i_2} min(t_{j_1}, t_{j_2})\right)^2 \tag{38}$$

$$M_2^{(3)}(0) = \frac{1}{n^2} \sum_{j_1,j_2=1}^{n} \sum_{i_1,i_2=1}^{p} F_{i_1}(0)F_{i_2}(0) \left(\sigma_{i_1}\sigma_{i_2}\rho_{i_1 i_2} min(t_{j_1}, t_{j_2})\right)^3 \tag{39}$$

Let's expand $g(z) = E\left[e^{i\theta X(z)}\right]$ until z^6 :

$$E\left[e^{i\theta X(z)}\right] = g(0) + g''(0)\frac{z^2}{2} + g^{(4)}(0)\frac{z^4}{24} + g^{(6)}(0)\frac{z^6}{720} + o(z^6) \tag{40}$$

where :

$$g(0) = e^{i\theta X(0)} \tag{41}$$

$$g''(0)\frac{z^2}{2} = g(0)\tilde{\theta} a_1(z) \tag{42}$$

$$g^{(4)}(0)\frac{z^4}{24} = g(0)\tilde{\theta}\left(-(i\theta-3)(i\theta-2)\frac{a_1^2(z)}{2} - (i\theta-2)b_1(z) - b_2(z)\right) \tag{43}$$

and

$$b_1(z) = \frac{z^4}{4A^3(0)} E\left[A'^2(0)A''(0)\right] \tag{44}$$

$$b_2(z) = \frac{z^4}{8A^2(0)} E\left[A''^2(0)\right] = a_1^2(z) - \frac{1}{2}a_2(z) \tag{45}$$

and, noting $\tilde{\rho}_{i_1 i_2} = \sigma_{i_1}\sigma_{i_2}\rho_{i_1 i_2}$ and $t_{j_1 j_2} = min(t_{j_1}, t_{j_2})$:

$$E\left[A'^2(0)A''(0)\right] = 2 \sum_{j_1,j_2,j_3=1}^{n} \sum_{i_1,i_2,i_3=1}^{p} F_{i_1}(0)F_{i_2}(0)F_{i_2}(0)F_{i_3}(0)\tilde{\rho}_{i_1 i_3}\tilde{\rho}_{i_1 i_3} t_{j_1 j_3} t_{j_2 j_3} \tag{46}$$

and, at last :

$$g^{(6)}(0)\frac{z^6}{720} = g(0)\tilde{\theta}((i\theta-5)(i\theta-4)(i\theta-3)(i\theta-2)\frac{a_1^3(z)}{6} \tag{47}$$

$$- (i\theta-4)(i\theta-3)(i\theta-2)c_1(z) \tag{48}$$

$$- (i\theta-3)(i\theta-2)c_1(z) - (i\theta-2)c_3(z) - c_4(z)) \tag{49}$$

where :

$$c_1(z) = \frac{z^6}{48A^5(0)}E\left[A'^4(0)A''(0)\right] = -a1(z)b1(z) \tag{50}$$

$$c_2(z) = \frac{z^6}{144A^4(0)}\left(9E\left[A'^2(0)A''^2(0)\right] + 4E\left(A'^3(0)A^{(3)}(0)\right)\right) \tag{51}$$

$$c_3(z) = \frac{z^6}{48A^3(0)}\left(4E\left[A'(0)A''(0)A^{(3)}(0)\right] + E\left[A''^3(0)\right]\right) \tag{52}$$

$$c_4(z) = \frac{z^6}{72A^2(0)}E\left[A^{(3)2}(0)\right] = a_1(z)a_2(z) - \frac{2}{3}a_1^3(z) - \frac{1}{6}a_3(z) \tag{53}$$

and noting $\tilde{A_{kl}} = \sum_{j=1}^n \sum_{i=1}^p F_i(0)\tilde{\rho}_{ik}t_{jl}$:

$$E\left[A'^2(0)A''^2(0)\right] = 8\sum_{j_1,j_2=1}^{n}\sum_{i_1,i_2=1}^{p} F_{i_1}(0)F_{i_2}(0)\tilde{A}_{i_1j_1}\tilde{A}_{i_2j_2}\tilde{\rho}_{i_1i_2}t_{j_1j_2} + 2M_2'(0)M_2''(0) \tag{54}$$

$$E\left[A'^3(0)A^{(3)}(0)\right] = 6\sum_{j=1}^{n}\sum_{i=1}^{p}F_i(0)\tilde{A}_{ij}^3 \tag{55}$$

$$E\left[A'(0)A''(0)A^{(3)}(0)\right] = 6\sum_{j_1,j_2=1}^{n}\sum_{i_1,i_2=1}^{p}F_{i_1}(0)F_{i_2}(0)\tilde{A}_{i_2j_2}(\tilde{\rho}_{i_1i_2}t_{j_1j_2})^2 \tag{56}$$

$$E\left[A''^3(0)\right] = 2\sum_{j_1,j_2,j_3=1}^{n}\sum_{i_1,i_2,i_3=1}^{p}F_{i_1}(0)F_{i_2}(0)F_{i_2}(0)F_{i_3}(0)\tilde{\rho}_{i_1i_3}\tilde{\rho}_{i_2i_3}\tilde{\rho}_{i_1i_2}t_{j_1j_3}t_{j_2j_3}t_{j_1j_2} \tag{57}$$

Multiplying the expansions of $\frac{1}{E\left[e^{i\theta Y(z)}\right]}$and $g(z) = E\left[e^{i\theta X(z)}\right]$ we obtain :

$$f(z) = 1 - i\theta d_1(z) - \theta^2 d_2(z) + i\theta^3 d_3(z) + \theta^4 d_4(z) + o(z^6) \tag{58}$$

where

$$
\begin{aligned}
d_1(z) =& 3a_1^2(z) + \frac{1}{2}a_2(z) - 2b_1(z) + b_2(z) - 20a_1^3(z) - \frac{1}{6}a_3(z) \\
& + 24c_1(z) - 6c_2(z) + 2c_3(z) - c_4(z)
\end{aligned} \tag{59}
$$

$$
\begin{aligned}
d_2(z) =& 5a_1^2(z) + \frac{1}{2}a_2(z) - 3b_1(z) + b_2(z) - \frac{128}{3}a_1^3(z) - \frac{1}{6}a_3(z) + 2a_1(z)b_1(z) \\
& - a_1(z)b_2(z) + 50c_1(z) - 11c_2(z) + 3c_3(z) - c_4(z)
\end{aligned} \tag{60}
$$

$$
d_3(z) = 2a_1^2(z) - b_1(z) - \frac{88}{3}a_1^3(z) + a_1(z)\left(5b_1(z) - 2b_2(z)\right) + 35c_1(z) - 6c_2(z) + c_3(z) \tag{61}
$$

$$
d_4(z) = -\frac{20}{3}a_1^3(z) + a_1(z)\left(-4b_1(z) + b_2(z)\right) - 10c_1(z) + c_2(z) \tag{62}
$$

so the approximation of the characteristic function of $X(1)$ is given by :

$$
E\left[e^{i\theta X(1)}\right] \approx e^{i\theta m(1) - \frac{1}{2}\theta^2 v(1)}\left(1 - i\theta d_1(1) - \theta^2 d_2(1) + i\theta^3 d_3(1) + \theta^4 d_4(1)\right) \tag{63}
$$

This induce the following approximation for the density function of $X(1)$:

$$
h(x) = p(x) + d_1(1)p'(x) + 2d_2(1)p''(x) + d_3(1)p^{(3)}(x) + d_4(1)p^{(4)}(x) \tag{64}
$$

where $p(x) = \frac{1}{\sqrt{2\pi v(1)}}e^{-\frac{(x-m(1))^2}{2v(1)}}$ is the density of $\mathcal{N}(m(1), v(1))$.

The price of the option is then approximate by :

$$
\begin{aligned}
V(0) &= e^{-rT}\int_{-\infty}^{+\infty}\left(e^x - K\right)^+ h(x)dx \\
&= e^{-rT}\left(I_0 + d_1(1)I_1 + d_2(1)I_2 + d_3(1)I_3 + d_4(1)I_4\right)
\end{aligned} \tag{65}
$$

where :

$$
I_0 = \int_{-\infty}^{+\infty}(e^x - K)^+ p(x)dx = e^{-rT}\left(M_1 N(y_1) - KN(y_2)\right) \tag{66}
$$

with $y_1 = \frac{m(1) - log(K)}{\sqrt{v(1)}} + \sqrt{v(1)}$ and $y_2 = y_1 - \sqrt{v(1)}$

$$
\begin{aligned}
I_1 &= \int_{-\infty}^{+\infty}(e^x - K)^+ p'(x)dx = \int_{log(K)}^{+\infty}(e^x - K)p'(x)dx \\
&= \left[(e^x - K)p(x)\right]_{log(K)}^{+\infty} - \int_{log(K)}^{+\infty} e^x p(x)dx = \int_{log(K)}^{+\infty} e^x p(x)dx
\end{aligned} \tag{67}
$$

and

$$
\begin{aligned}
I_2 &= \int_{log(K)}^{+\infty} (e^x - K)p''(x)dx = [(e^x - K)p'(x)]_{log(K)}^{+\infty} - \int_{log(K)}^{+\infty} e^x p'(x)dx \\
&= -I_1 - Kp(log(K))
\end{aligned}
\tag{68}
$$

and, still using by part integration :

$$
I_3 = -I_2 - Kp'(log(K)) \tag{69}
$$

$$
I_4 = -I_3 - Kp''(log(K)) \tag{70}
$$

Finally, using the fact that $d_1(1) - d_2(1) + d_3(1) - d_4(1) = 0$ ($\theta = -i$ in equation 58) we obtain :

$$
V(0) = e^{-rT} \left(M_1 N(y_1) - KN(y_2) + K(z_1 p(log(K)) + z_2 p'(log(K)) + z_3 p''(log(K))) \right) \tag{71}
$$

where

$$
\begin{aligned}
z_1 &= d_2(1) - d_3(1) + d_4(1) && (72)\\
z_2 &= d_3(1) - d_4(1) && (73)\\
z_3 &= d_4(1) && (74)
\end{aligned}
$$

Note that the first terms correspond to the Levy approximation. So Ju adds a correction term to the Levy approximation.

In term of performance, the complexity of the algorithm is $O\left((np)^3\right)$

2.6 Curran

Curran method is based on conditioning. It is often called a conditional expectation method. The basic idea is to separate the price of the call in two parts by conditioning with the geometric average. One part can be evaluated "exactly", the other part is approximated using a moment matching method. The method presented hereafter for the approximated part is not the original method of Curran, but an enhanced one.

Let $G(T)$ be the geometric average :

$$G(T) = \prod_{j=1}^{n} \prod_{i=1}^{p} \left(\frac{1}{n\alpha_i} F_i(t_j) \right)^{\alpha_i} \tag{75}$$

where :

$$\alpha_i = \frac{\frac{1}{n} F_i(0)}{\sum_{i=1}^{p} F_i(0)} \tag{76}$$

The convexity of the exponential function assures, by the Jensen inequality, that :

$$G(T) = exp \left(\sum_{j=1}^{n} \sum_{i=1}^{p} \alpha_i log(\frac{1}{n\alpha_i} F_i(t_j)) \right) \leq \alpha_i \sum_{j=1}^{n} \sum_{i=1}^{p} exp \left(log(\frac{1}{n\alpha_i} F_i(t_j)) \right) = A(T) \tag{77}$$

Then, conditioning by the geometric average we can separate the price of a call in two parts :

$$\begin{aligned} V(0) &= e^{-rT} E\left[(A(T)-K)^+\right] \\ &= e^{-rT}\left(E[(A(T)-K)^+ | G(T) < K] + E[(A(T)-K)^+ | G(T) \geq K]\right) \end{aligned} \tag{78}$$

We will at first deal with the "exact" part (we use the property $A(T) > G(T)$) :

$$E[(A(T)-K)^+ | G(T) \geq K] = E[A(T) | G(T) \geq K] - K\mathbb{P}(G(T) \geq K) \tag{79}$$

Let's note $m_G = E[log(G(T))]$, $\sigma_G = \frac{1}{T}\sqrt{Var[log(G(T))]}$ and $cov_i(t) = Cov[log(G(T)), log(F_i(t))]$. $(log(G(T)), log(F_i(t)))$ being by hypothesis bivariate normally distributed. The conditional distribution of $log(F_i(t))$ given $log(G(T))$ is normally distributed :

$$log(F_i(t)) | log(G(T)) \sim \mathcal{N}\left(ln(F_i(0)) - \frac{1}{2}\sigma_i^2 t + \frac{cov_i(t)}{\sigma_G^2 T}(log(G(T)) - m_G), \sigma_i^2 t - \frac{cov_i^2(t)}{\sigma_G^2 T} \right) \tag{80}$$

So :

$$
\begin{aligned}
&E[F_i(t_j)|G(T) \geq K] \\
&= \int_K^{+\infty} F_i(0) exp\left(\frac{cov_i(t_j)}{\sigma_G^2 T}\left(log(g) - m_G - \frac{1}{2}cov_i(t_j)\right)\right)\frac{exp\left(-\frac{(log(g)-m_G)^2}{2\sigma_G^2 T}\right)}{g\sigma_G\sqrt{2\Pi T}} dg \\
&= \int_K^{+\infty} F_i(0)\frac{1}{g\sigma_G\sqrt{2\Pi T}} exp\left(-\frac{(cov_i(t_j) - (log(g) - m_G))^2}{2\sigma_G^2 T}\right) dg \\
&= F_i(0)\mathcal{N}\left(\frac{cov_i(t_j) - (log(K) - m_G)}{\sigma_G\sqrt{T}}\right)
\end{aligned}
\tag{81}
$$

And, we finally get for the "exact" part :

$$
\begin{aligned}
E[(A(T) - K)^+|G(T) \geq K] =& \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{p} F_i(0)\mathcal{N}\left(\frac{cov_i(t_j) - (log(K) - m_G)}{\sigma_G\sqrt{T}}\right) \\
&- K\mathcal{N}\left(\frac{m_G - log(K)}{\sigma_G\sqrt{T}}\right)
\end{aligned}
\tag{82}
$$

where :

$$
m_G = log(A(0)) - \sum_{j=1}^{n}\sum_{i=1}^{p}\frac{1}{2}\alpha_i\sigma_i^2 t_j \tag{83}
$$

and

$$
\sigma_G^2 T = \sum_{j_1,j_2=1}^{n}\sum_{i_1,i_2=1}^{p}\alpha_{i_1}\alpha_{i_2}\sigma_{i_1}\sigma_{i_2}\rho_{i_1 i_2} min(t_{j_1}, t_{j_2}) \tag{84}
$$

and

$$
cov_i(t_j) = \sum_{j_1=1}^{n}\sum_{i_1=1}^{p}\alpha_{i_1}\sigma_i\sigma_{i_1}\rho_{i i_1} min(t_j, t_{j_1}) \tag{85}
$$

Let's now concentrate on the approximated part :

$$
E[(A(T)-K)^+|G(T) \leq K] = \int_0^K E[(A(T)-K)^+|G(T) = g]dF_{G(T)}(g) \tag{86}
$$

we will approximate the distribution of $H_g = ((A(T) - g)|G(T) = g)$ by a lognormal distribution using the moment matching method of Levy:

$$
H_g \approx M_H exp(-\frac{1}{2}\sigma_H^2 T + \sigma_H W_T) \tag{87}
$$

where :

$$M_H = E[A(T) - g|G(T) = g] = \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{p} E[F_i(t_j)|G(T) = g] - g \tag{88}$$

and

$$\sigma_H^2 T = log\left(E[H_g^2]\right) - 2log(M_H) \tag{89}$$

and

$$\begin{aligned} E[H_g^2] &= E[A^2(T)|G(T) = g] - 2gE[A(T)|G(T) = g] + g^2 \\ &= \frac{1}{n}\sum_{j_1,j_2=1}^{n}\sum_{i_1,i_2=1}^{p} E\left[F_{i_1}(t_{j_1})F_{i_2}(t_{j_2})|G(T) = g\right] - 2gE[A(T)|G(T) = g] + g^2 \end{aligned} \tag{90}$$

where, we deduct from the distribution of $log(F_i(t))$ given $log(G(T))$ that
:

$$\begin{aligned} E[F_i(t_j)|G(T) = g] &= F_i(0)exp\left(-\frac{1}{2}\sigma_i^2 t_j + \frac{cov_i(t_j)}{\sigma_G^2 T}(log(g) - m_G) + \frac{1}{2}\left(\sigma_i^2 t_j - \frac{cov_i^2(t_j)}{\sigma_G^2 T}\right)\right) \\ &= F_i(0)exp\left(\frac{cov_i(t_j)}{\sigma_G^2 T}\left(log(g) - m_G - \frac{1}{2}cov_i(t_j)\right)\right) \end{aligned} \tag{91}$$

and noting $cov_{i_1 i_2}(t_{j_1}, t_{j_2}) = Cov\left[log(G(T)), log(F_{i_1}(t_{j_1})F_{i_2}(t_{j_2}))\right]$:

$$\begin{aligned} &E[F_{i_1}(t_{j_1})F_{i_2}(t_{j_2})|G(T) = g] = \\ &F_{i_1}(0)F_{i_2}(0)exp\left(\begin{array}{l} -\frac{1}{2}\sigma_{i_1}^2 t_{j_1} - \frac{1}{2}\sigma_{i_2}^2 t_{j_2} + \frac{cov_{i_1 i_2}(t_{j_1},t_{j_2})}{\sigma_G^2 T}(log(g) - m_G) \\ +\frac{1}{2}\left(\sigma_{i_1}^2 t_{j_1} + \sigma_{i_2}^2 t_{j_2} + 2\rho_{i_1 i_2}\sigma_{i_1}\sigma_{i_2} min(t_{j_1}, t_{j_2}) - \frac{cov_{i_1 i_2}^2(t_{j_1},tj_2)}{\sigma_G^2 T}\right) \end{array}\right) \\ &= F_{i_1}(0)F_{i_2}(0)exp\left(\frac{cov_{i_1 i_2}(t_{j_1}, t_{j_2})}{\sigma_G^2 T}\left(log(g) - m_G - \frac{1}{2}cov_{i_1 i_2}(t_{j_1}, t_{j_2})\right) + \rho_{i_1 i_2}\sigma_{i_1}\sigma_{i_2} min(t_{j_1}, t_{j_2})\right) \end{aligned} \tag{92}$$

where

$$cov_{i_1 i_2}(t_{j_1}, t_{j_2}) = cov_{i_1}(t_{j_1}) + cov_{i_2}(t_{j_2}) \tag{93}$$

Thus :

$$E[(A(T) - K)^+|G(T) = g] \approx M_H \mathcal{N}(d_1(g)) - (K - g)\mathcal{N}(d_2(g)) \tag{94}$$

where

$$d_1(g) = \frac{log(M_H) - log(K-g)}{\sigma_H\sqrt{T}} + \frac{1}{2}\sigma_H\sqrt{T} \tag{95}$$

and

$$d_2(g) = d_1(g) - \sigma_H\sqrt{T} \tag{96}$$

This expression still has to be integrated as the approximated part we are dealing with is :

$$E[(A(T)-K)^+|G(T) \leq K] = \int_0^K E[(A(T)-K)^+|G(T) = g]dF_{G(T)}(g) \tag{97}$$

where

$$dF_{G(T)}(g) = \frac{1}{g\sigma_G\sqrt{2\pi T}}exp\left(-\frac{(log(g)-m_G)^2}{2\sigma_G^2 T}\right)dg \tag{98}$$

The integral has to be computed by means of numerical quadrature. Of course, the integrand has to be evaluated on each integration point.

Note : In the original method of Curran, it is remarked that the main part of the integral comes from the value of geometric mean close to the strike K. Then he makes the following approximation :

$$\int_0^K E[(A(T)-K)^+|G(T) = g]dF_{G(T)}(g) = E[(A(T)-K)^+|G(T) = K]\mathbb{P}\left(G(T) < K\right) \tag{99}$$

Thus, there is no need to implement numerical quadrature, but it is well known that this approximation is not very accurate out of the money.

Note : in a first step we used the geometric average :

$$G(T) = \left(\prod_{j=1}^{n}\prod_{i=1}^{p} pF_i(t_j)\right)^{\frac{1}{pn}} \tag{100}$$

But we changed it as the one we use is closer to the geometric average and so gives exact part closer to the value of the option and better results on the price of the option.

3 Experiments

In this section we compare the accuracy of the different methods previously described.

The first goal is to evaluate the precision of the models in different reasonable situations. For this purpose, we tested the models with

- different strikes,
- different maturities,
- different volatilities and
- different numbers of fixing date.

Another goal of the study, which was conducted as an internship in company Sophis, was to study the conditions in which the company's algorithm found acceptable prices.

Finally, note that the ultimate goal is not only precision, since the Monte-Carlo simulation could be used for that purpose. The important aspect is the trade-off between prices precision and the algorithm performance. So we also conducted some experiments on the performance of the different models. However, it should be noted that it is very difficult to draw definite conclusion in terms of performance since the architecture of a real application has a significant impact on the speed. For example the use of parallel computing can reduce the computation time from the users point of view. In our study, we compare the relative performance of the models. We compare performances of the studied methods for different numbers of fixing date.

The asian basket option that we choose as reference for all our tests has two underlyings and the following properties :

$F_1(0) = 100$, $F_2(0) = 80$, $\sigma_1 = \sigma_2 = 0.4$, $T = 2$, $\rho_{12} = 0.5$, $K = 180$, $n = 10$, $t_j = j\frac{T}{n}$.

Thus, this reference option, can be considered as a usual option:

- close price of the elements of the basket
- 40% a year volatility

- positive correlation
- at the money strike
- 2 years maturity

The only unusual value is the number of averaging date (n=10). It has been chosen small for question of performance.

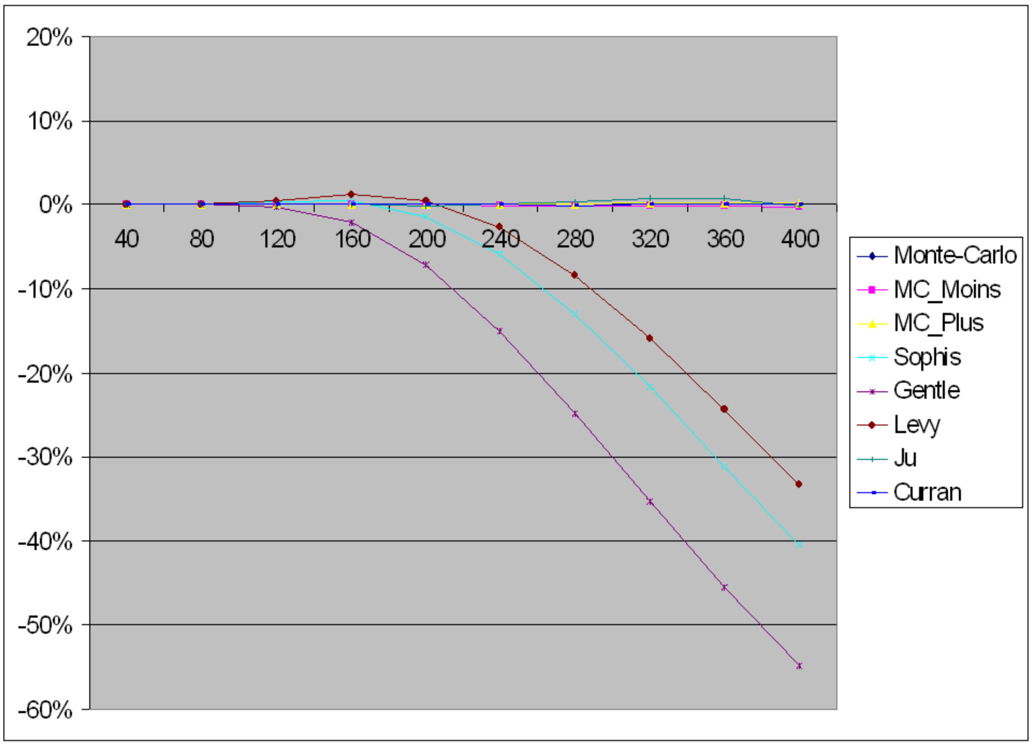
Starting with this option, we study the impact of each parameter on the accuracy of the prices obtained by the five studied methods. To test the accuracy of these prices we need a reference price. To compute this price, we use monte-carlo simulations (4 millions). We implemented antithetic control, importance sampling and control variate to reduce the variance of monte-carlo simulations and, so, optimize the accuracy of the reference price. We use a Gauss-Legendre integration of order 50 to compute the approximated part of Curran method.

Graphs present the price obtained by the models described in first part, and the bounds of the 95% confidence interval obtained by Monte-Carlo simulations. The reference price is the middle of the confidence interval. Prices are represented by their difference to the reference price expressed in percentage of the reference price.

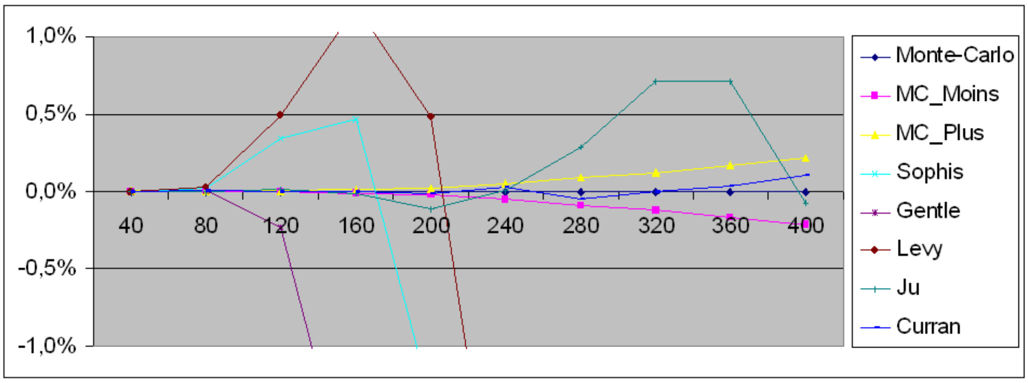
3.1 sensitivity to strike

The number of Monte-Carlo simulations is set to 10 millions to obtain an acceptable accuracy for out of the money strikes.

Strike varies from 40 to 400 by step of 40 (money is at 180).



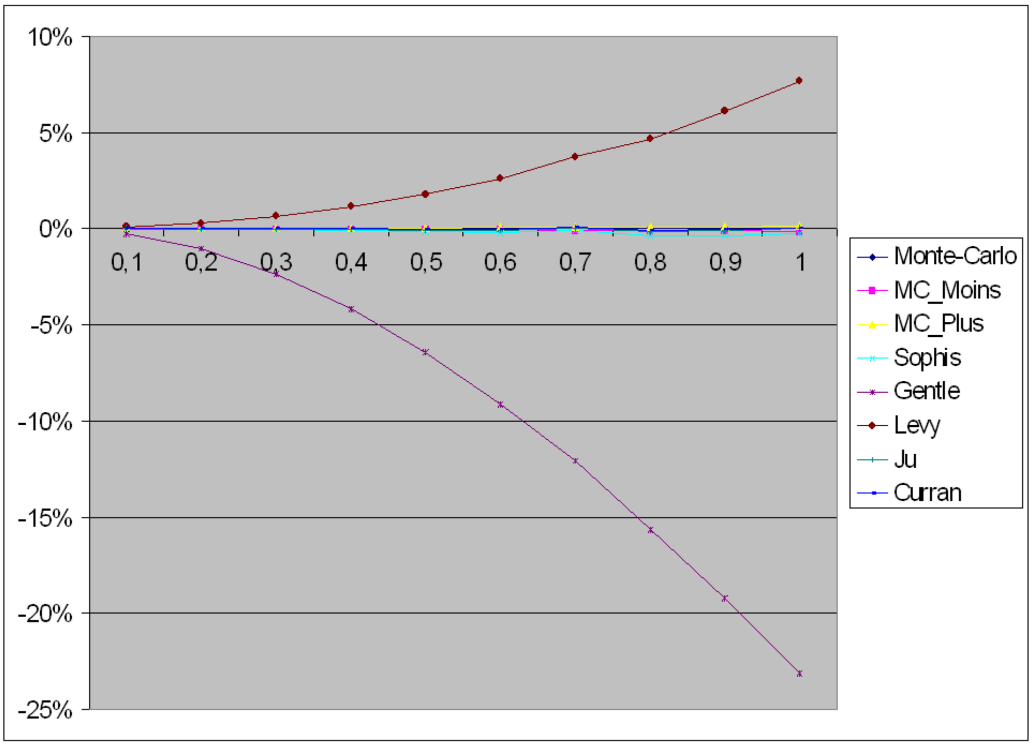
All models give acceptable price in the money. Prices obtained by Gentle, Sophis and Levy diverge strongly out of the money.



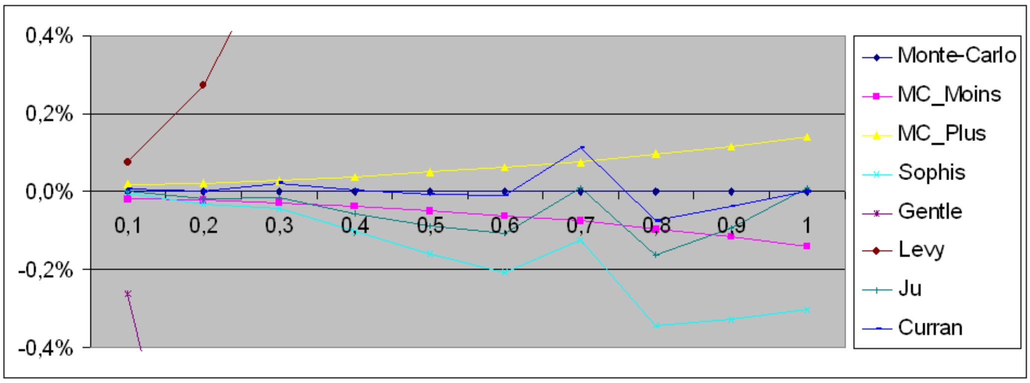
Curran gives the best accuracy far out of the money. Prices obtained par Ju keep acceptable.

3.2 sensitivity to volatility

Volatility varies from 0.1 to 1 by step of 1.



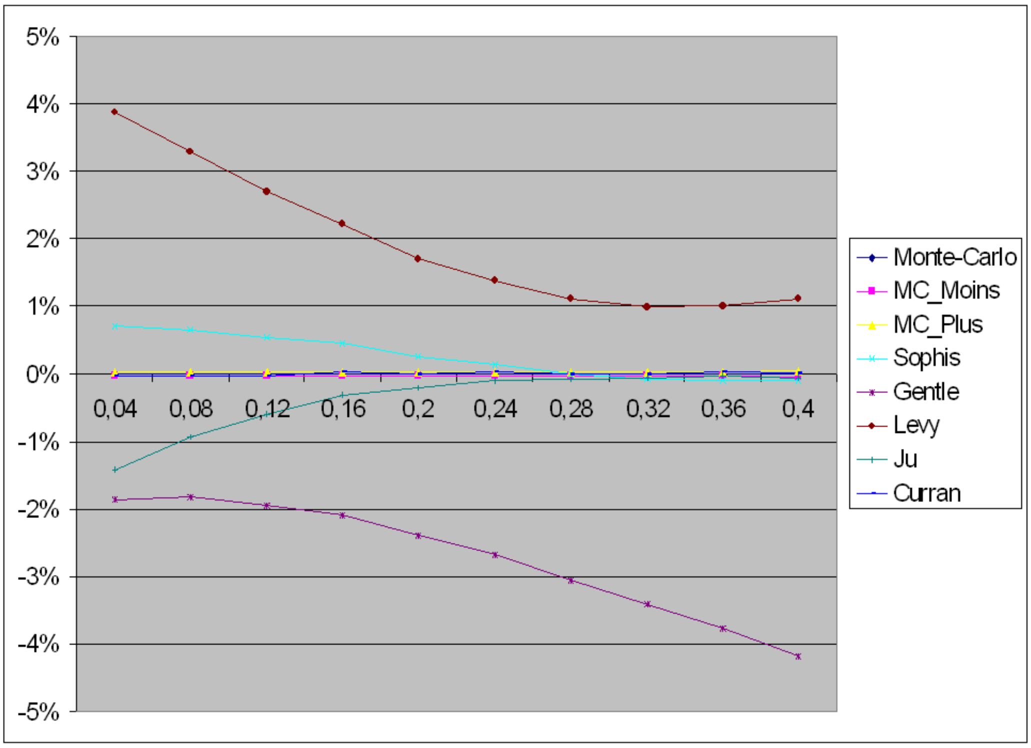
Prices obtained by Gentle and Levy diverge when volatility rise.



Prices obtained by Curran, Ju and Sophis keep very accurate when volatility rise.

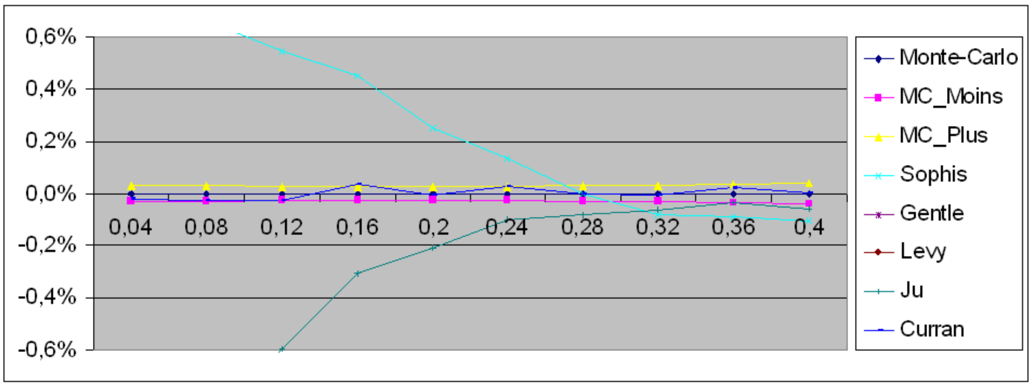
3.3 sensitivity to relative volatility

Volatility of F_1 varies from 0.04 to 0.4 by step of 0.04 while volatility of F_2 stays at 0.4.



Gentle is the only model giving better prices when the difference between the volatilities of F_1 and F_2 grows. Prices obtained by Ju and Sophis loose precision when the difference between the volatilities of F_1 and F_2 grows, but they keep acceptable.

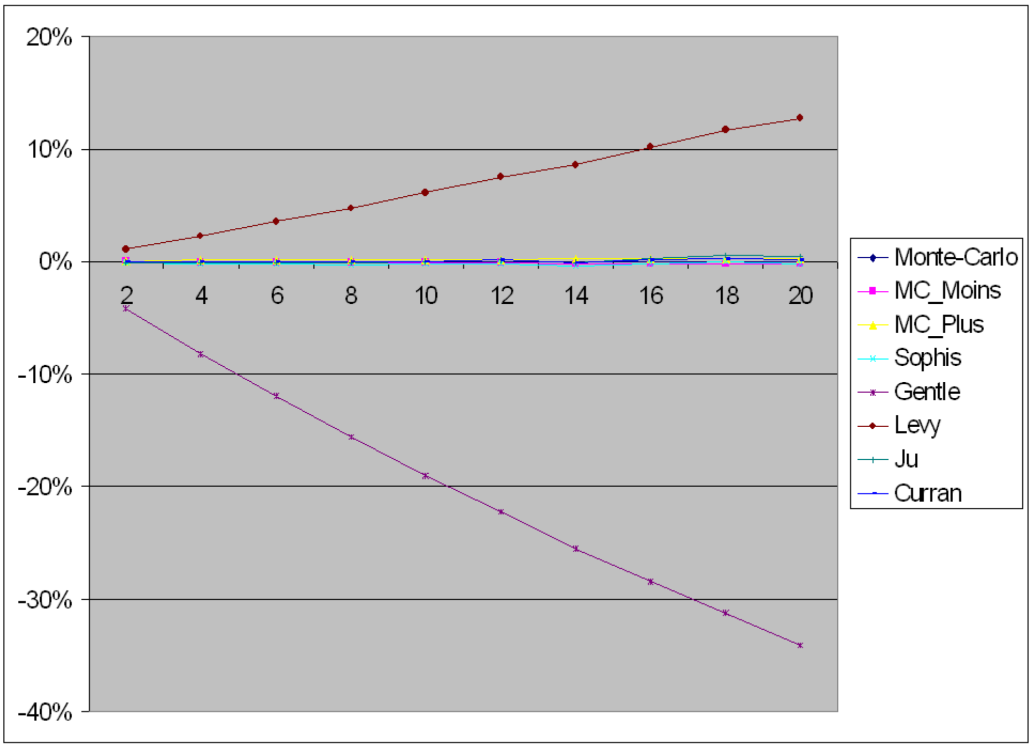
Curran gives the most accurate prices.



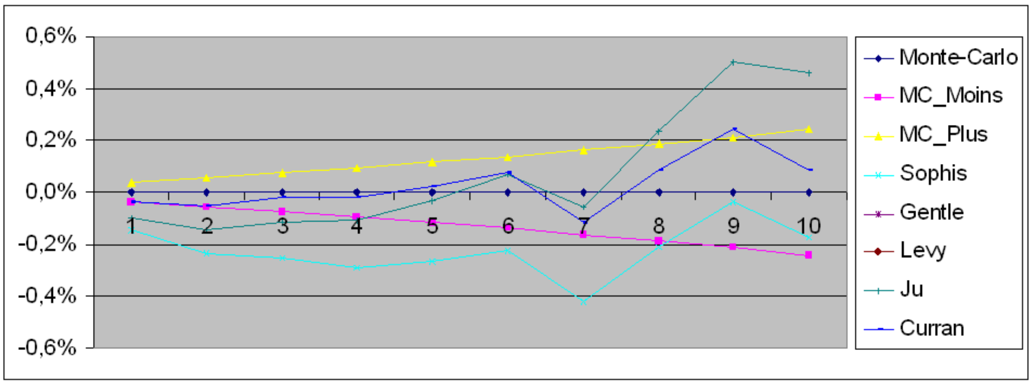
3.4 sensitivity to Maturity

Sensitivity to maturity is of course the same as sensitivity to volatility as, in the conditions of our experimentation, the price of the option is function of $\sigma_i\sqrt(T)$. But, it seems interesting to see the result for a "realistic" range of maturity as we presented it for a realistic range of volatility.

Maturity varies from 2 to 20 by step of 2.

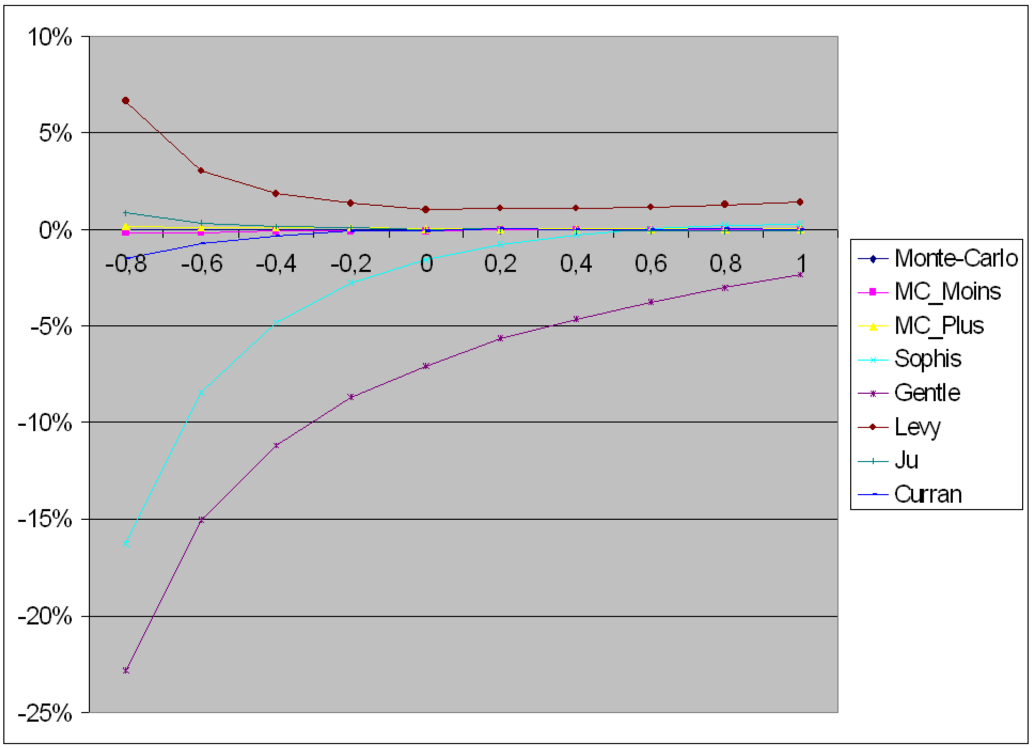


Unsurprisingly, as for volatility, prices obtained by Gentle and Levy diverge when maturity rises. Prices obtained by Sophis, Ju and Curran keeps very accurate.

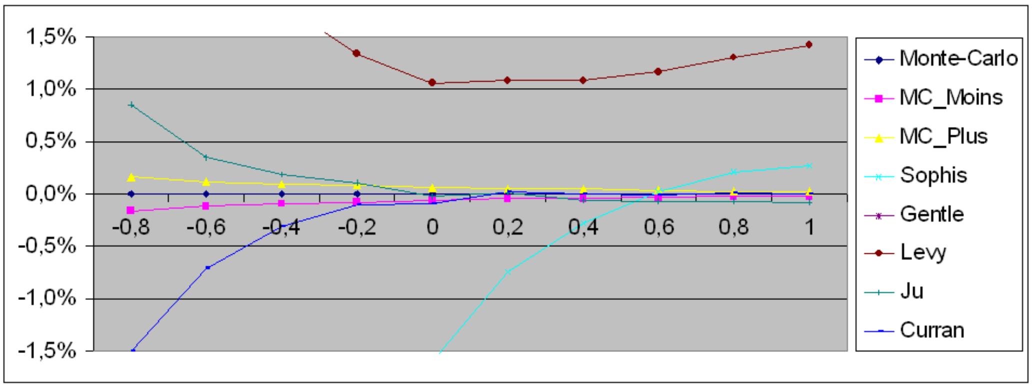


3.5 sensitivity to correlation

Correlation varies from -0.8 to 1 by step of 0.2.



Prices given by all our models diverge when correlation decrease. Prices obtained by Curran and Ju keeps acceptable.

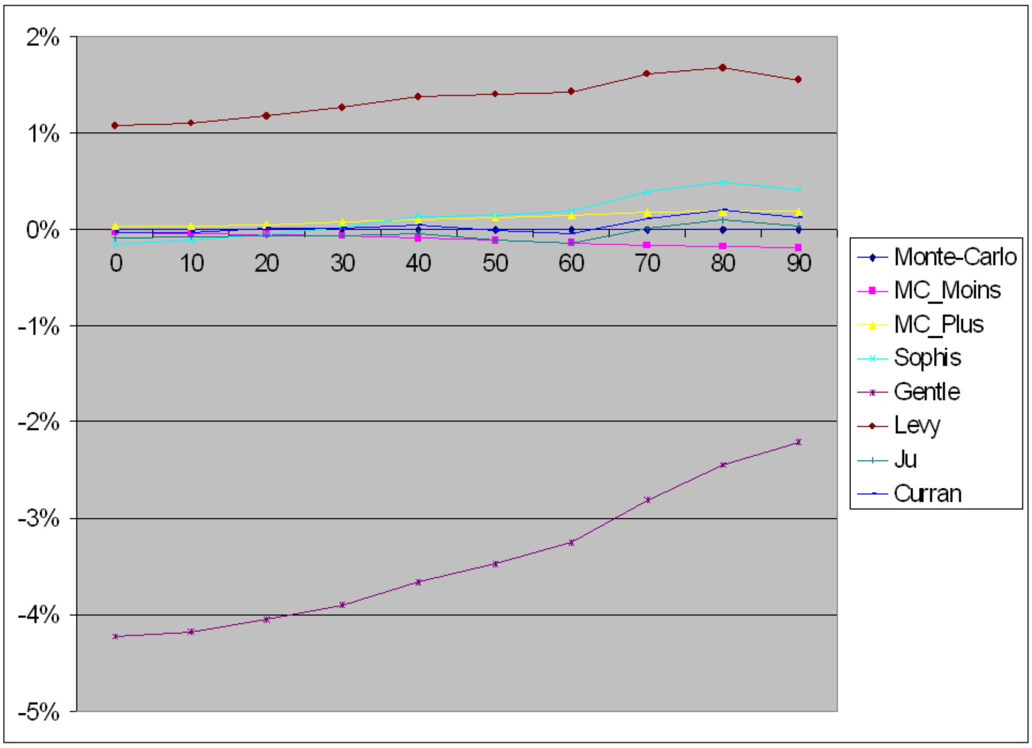


One may be surprised that, for negative correlation Curran underestimate the price when Levy overestimate the price as Curran is based on an exact part plus an approximated part based on Levy approach. The point is that the Levy approximation used for the computation of Curran deals with out of the money option. As we can see in the "sensitivity to strike" section, Levy strongly underestimate the price of this kind of option. Moreover when the correlation decrease from 1 to -0.8, the share of the approximated part in the Curran price increase from 1% to 30%.

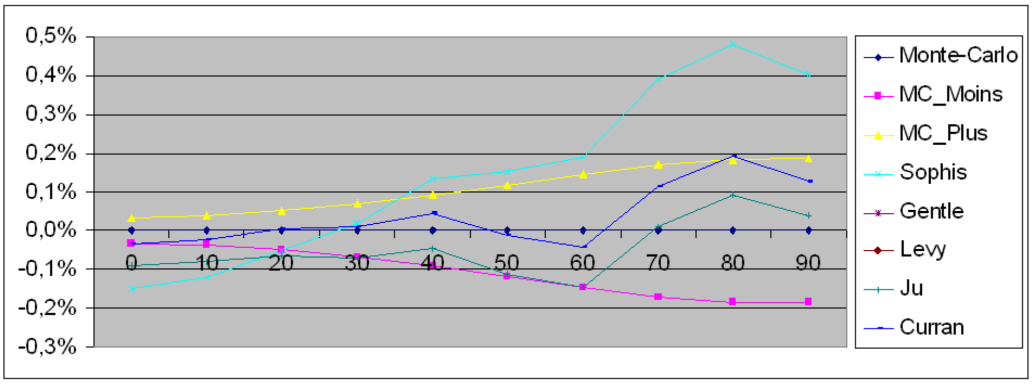
3.6 sensitivity to relative spot

In this section we modify the spot of both Future prices but we keep their sum constant.

$F_1(0) = 90.1 + x$ and $F_2(0) = 90.1 - x$ where x varies from 0 to 90 by step of 10.

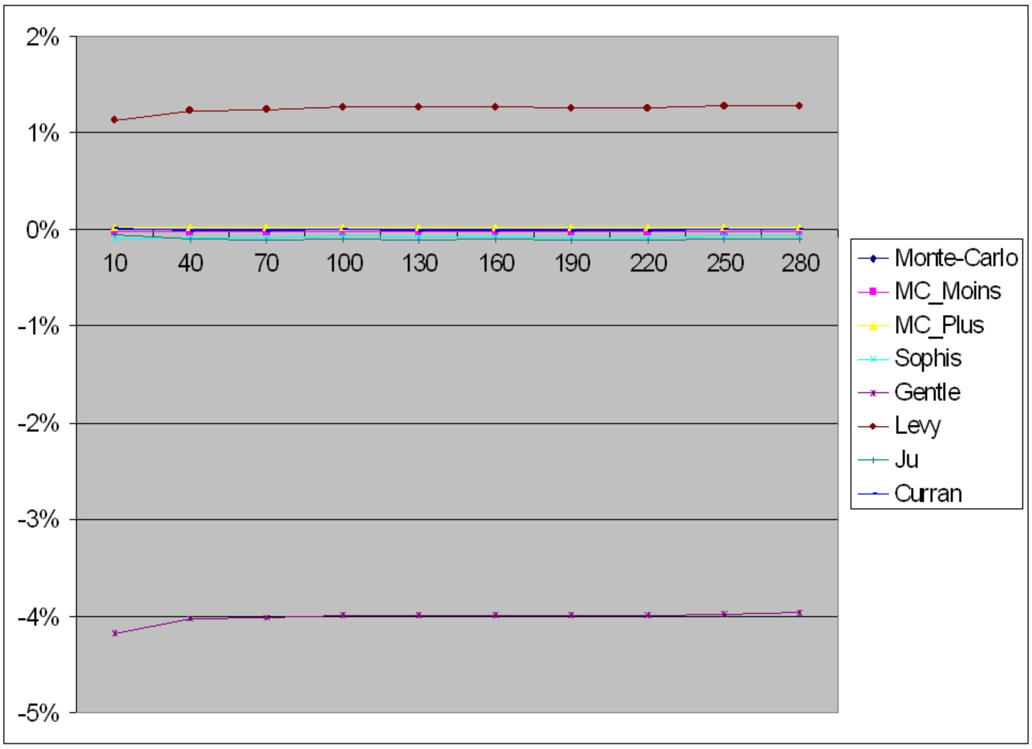


Once more, most accurate prices are given by Ju and Curran.

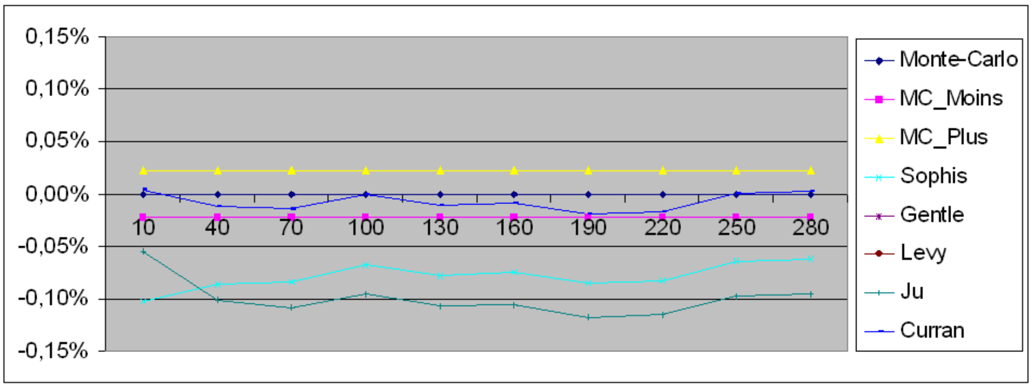


3.7 sensitivity to number of fixing dates

The number of fixing dates varies from 10 to 100 by step of 10



For the range experimented, there is no notable effect of the number of fixing dates on the accuracy of the prices given by tested models.



3.8 performance

The following array contains the prices computing for each model for a number of time interval varying from 10 to 460 by step of 50. Time is expressed in milliseconds.

Period number	Monte-Carlo	Sophis	Gentle	Levy	Ju	Curran
10	31249	0	0	0	0	0
60	175334	0	0	0	343	203
110	319419	0	0	0	2062	672
160	463926	0	0	16	6296	1407
210	607996	0	16	16	14155	2406
260	752550	16	0	16	26826	3672
310	896558	0	0	15	45435	5203
360	1041471	0	0	31	71074	7031
410	1184947	16	0	47	105197	9094
460	1329142	15	0	63	148413	11421

The computation time is obtained with the C function clock which is supposed to provide a precision of 1 millisecond. But the results of the experiment shows that it is not the case. We tried to use the Windows specific function GetProcessTimes which is supposed to provide a precision of 1 nanosecond and take into account only the time dedicated by the processor to our process. But the precision of the result was even worse. However, we are mainly interested by the relative results of the different pricing methods and we got it.

Unsurprisingly, computation time rises linearly for Monte-carlo and exponentially for Curran and Ju. For 460 period of times (ie about 2 years daily average), computation time by Curran keeps acceptable (11.5 seconds) while computation time by Ju becomes problematic (more than 2 minutes). The best performers are, of course, the 3 basic models. Levy seems to be slightly slower than its two contestants, may be due to the presence of log in the volatility computation formula.

3.9 conclusion

The following draw resumes the results of the experimentation :

	Sophis	Gentle	Levy	Curran	Ju
strike	- -	- -	- -	++	++
volatility	++	- -	0	++	++
relative spot	++	0	+	++	++
Maturity	++	- -	-	++	++
Relative vol	++	0	0	++	+
Correlation	- -	- -	-	+	++
Period	++	0	+	++	++

Where the symbols +, 0 and - represents the difference between the price obtained by a model and the closest value of confidence interval obtained by Monte-carlo simulation :

- - stands for greater than 10%

- stands for between 5% and 10%

0 stands for between 2% and 5%

+ stands for between 1% and 2%

++ stands for lower than 1%

The model chosen by Sophis gives the most accurate price in the category "naive log-normal distribution approach". In case of positive correlations it gives acceptable prices for in the money options. But, to get acceptable prices for out of the money options and negative correlations we need to use the more sophisticated models proposed by Curran and Ju.

Beyond its great accuracy, the main quality of Ju model is it simplicity of implementation. Of course, formulas are long but they just have to be applied.

Curran method is more complex to implement. It requires a numerical integration for the approximated part. The risk is then to loose the accuracy of the theoretical approximation by numerical integration error.

Performance gives a decisive advantage to Curran compared to Ju. For similar precision, Curran complexity is in $O\left((np)^2\right)$ while Ju complexity is in $O\left((np)^3\right)$. This being confirmed by our experimentation.

4 Enhancements of optimized-Deltas in Sophis Model

The price of an option is defined as the amount necessary to hedge the option. Once a trader has fixed the price of an option for a deal, he will need the deltas of the option for hedging purposes.

The first and universal idea to compute deltas is to use finite differences. But, as shown further, this method has poor performances. So, Sophis implemented an optimized deltas computation in its software. This method generates only partial deltas (ie deltas that do not integrate the dependance of the volatility on the spot of the underlyings). As traders using sophis require total deltas we have completed the optimized partial deltas method to compute optimized total deltas.

In this section, we will first describe how asian option are modeled in Sophis as it covers a wider framework that the one previously studied. Then we will describe optimized total deltas computation. We have implemented optimized total deltas computation in Sophis software, and so, we will at last validate it.

4.1 Asian options on basket of commodities in Sophis

There are two main differences between general asian options on basket treated by Sophis and asian options on basket studied in the first section of this document.

- different commodities in single basket may be traded in different currencies. So terms of Foreign Exchange between commodity currencies and basket currency appear.
- Fixing dates on which means are computed may differ from commodity to commodity and from commodity to forex.

That leads to express the underlying of a call this way :

$$A(T) = \sum_{i,t_k,t_l} u_i(t_k, t_l)\chi_i(t_l)F_i(t_k) \tag{101}$$

where

- F_i is a commodity future.

- χ_i is the change rate between the currency of F_i and the currency of the option.

- u_i express a weight. It can be equal to 0 for some (t_k, t_l).

- dates t_k and t_l are taken in the possible quotation date for futures or Forex.

There is a special treatment for precious metals, they are considered as currencies. So they are integrated in the basket as a Future with fixed value 1 and a forex between the precious metal and the currency of the option.

Let's now place at date 0 :

$$A(0) = \sum_{i,t_k,t_l} u_i(t_k, t_l)\chi_i^0(t_l)F_i(0) \tag{102}$$

where $\chi_i^0(t_l)$ is the forex forward.

The geometric average introduced for the computation of the volatility in subsection 2.2 is then :

$$G(T) = \prod_{i,t_k,t_l} \left(\frac{u_i(t_k, t_l)}{\alpha_i(t_k, t_l)}\chi_i(t_l)F_i(t_k)\right)^{\alpha_i(t_k,t_l)} \tag{103}$$

$$= \prod_{i,t_k,t_l} \left(\frac{u_i(t_k, t_l)}{\alpha_i(t_k, t_l)}\right)^{\alpha_i(t_k,t_l)} \prod_{i,t_l} (\chi_i(t_l))^{\gamma_i(t_l)} \prod_{i,t_k} (F_i(t_k))^{\beta_i(t_k)} \tag{104}$$

where

$$\alpha_i(t_k, t_l) = \frac{u_i(t_k, t_l)\chi_i^0(t_l)F_i(0)}{A(0)} \tag{105}$$

and

$$\gamma_i(t_l) = \sum_{t_k} \alpha_i(t_k, t_l) \tag{106}$$

and

$$\beta_i(t_k) = \sum_{t_l} \alpha(t_k, t_l) \tag{107}$$

So the volatility of the geometric average is given by :

$$\begin{aligned}\sigma^2 T = &\sum_{i_1,i_2,t_{k_1},t_{k_2}} \beta_{i_1}(t_{k_1})\beta_{i_2}(t_{k_2})\rho_{F_{i_1},F_{i_2}}\sigma_{F_{i_1}}(t_{k_1})\sigma_{F_{i_2}}(t_{k_2})min(t_{k_1},t_{k_2}) \\ &+ \sum_{i_1,i_2,t_{l_1},t_{l_2}} \gamma_{i_1}(t_{l_1})\gamma_{i_2}(t_{l_2})\rho_{\chi_{i_1},\chi_{i_2}}\sigma_{\chi_{i_1}}(t_{l_1})\sigma_{\chi_{i_2}}(t_{l_2})min(t_{l_1},t_{l_2}) \\ &+ 2\sum_{i_1,i_2,t_k,t_l} \gamma_{i_1}(t_l)\beta_{i_2}(t_k)\rho_{\chi_{i_1},F_{i_2}}\sigma_{\chi_{i_1}}(t_l)\sigma_{F_{i_2}}(t_k)min(t_k,t_l)\end{aligned} \tag{108}$$

The strike for volatility allowing to extract volatilities from the volatility surface is chosen to be :

- $K_{F_i} = \frac{K}{A(0)}F_i(0)$ for commodities
- $K_{\chi_i,t} = \frac{K}{A(0)}\chi_i^0(t)$ for precious metal
- $K_{\chi_i,t} = \chi_i^0(t)$ for forex

The correlation between forex are deducted from forex volatilities.

We can note that the complexity of the computation is in $O(n^2p^2)$ where n is the number of future and p is the number of fixing date.

Deltas are computed with respect to each risk sources (Future and Forex). The first idea is to compute Delta by finite differences, but it implies to compute volatility twice for each risk sources. As we may have $2n$ risk sources (n futures and n forex), the complexity of the computation of the deltas is in $O(4n^3p^2)$.

To improve performance, we can think about decomposing deltas this way :

$$\Delta F_i = \frac{\partial P}{\partial A(0)}\frac{\partial A(0)}{\partial F_i}$$
$$\Delta \chi_i = \frac{\partial P}{\partial A(0)}\frac{\partial A(0)}{\partial \chi_i}$$

where

$$\frac{\partial A(0)}{\partial F_i} = \sum_{t_k,t_l} u_i(t_k,t_l)\chi_i(t_l)$$
$$\frac{\partial A(0)}{\partial \chi_i} = \sum_{j|C_j=C_i,t_k,t_l} u_i(t_k,t_l)F_i(t_k)\frac{\chi_j^0(t_l)}{\chi_j(0)}$$

and $\frac{\partial P}{\partial A(0)}$ is computed by finite differences. Then we have to recompute the volatility only twice (for the evaluation of $\frac{\partial P}{\partial A(0)}$). C_i represents the currency in which F_i is quoted.

Unfortunately, this decomposition is not correct. It gives only a partial delta. The point is that the price of the option does not only depends on risk sources through $A(0)$ but although through the volatility.

4.2 Total Delta Future

Total deltas with respect to futures are given by :

$$\Delta F_i = \frac{\partial P}{\partial A(0)} \frac{\partial A(0)}{\partial F_i} + \frac{\partial P}{\partial \sigma} \frac{\partial \sigma}{\partial F_i} \tag{109}$$

where :

$$\frac{\partial \sigma}{\partial F_i} = \frac{1}{2\sigma} \frac{\partial \sigma^2}{\partial F_i} = \frac{1}{2\sigma} \frac{1}{A(0)^2} \left(\frac{\partial A(0)^2 \sigma^2}{\partial F_i} - \sigma^2 \frac{\partial A(0)^2}{\partial F_i} \right) \tag{110}$$

This decomposition will simplify calculation as $A(0)\beta_i$ and $A(0)\gamma_i$ are linear functions of F_i.

$$\frac{\partial A(0)^2}{\partial F_i} = 2A(0) \sum_{t_k, tl} u_i(t_k, t_l) \chi_i(t_l) \tag{111}$$

and, as σ depends on F_i trough β_i, γ_i, the volatility of each future and of each precious metal :

$$\begin{aligned} \frac{\partial A(0)^2 \sigma^2}{\partial F_i} &= \sum_{t_l} \frac{\partial A(0)^2 \sigma^2}{\partial A(0)\gamma_i(t_l)} \frac{\partial A(0)\gamma_i(t_l)}{\partial F_i} + \sum_{t_k} \frac{\partial A(0)^2 \sigma^2}{\partial A(0)\beta_i(t_k)} \frac{\partial A(0)\beta_i(t_k)}{\partial F_i} \\ &+ \sum_{j,t_k} \frac{\partial A(0)^2 \sigma^2}{\partial \sigma_{F_j}(t_k)} \frac{\partial \sigma_{F_j}(t_k)}{\partial F_i} + \sum_{j,t_l} \frac{\partial A(0)^2 \sigma^2}{\partial \sigma_{\chi_j}(t_l)} \frac{\partial \sigma_{\chi_j}(t_l)}{\partial F_i} \end{aligned} \tag{112}$$

and

$$\begin{aligned} T \frac{\partial A(0)^2 \sigma^2}{\partial A(0)\gamma_i(t_l)} &= 2A(0) \sum_{i_2, t_{l_2}} \rho_{\chi_i, \chi_{i_2}} \sigma_{\chi_i}(t_l) \sigma_{\chi_{i_2}}(t_{l_2}) min(t_l, t_{l_2}) \gamma_{i_2}(t_{l_2}) \\ &+ 2A(0) \sum_{i_2, t_k} \rho_{\chi_i, F_{i_2}} \sigma_{\chi_i}(t_l) \sigma_{F_{i_2}}(t_k) min(t_l, t_k) \beta_{i_2}(t_k) \end{aligned} \tag{113}$$

and

$$T\frac{\partial A(0)^2\sigma^2}{\partial A(0)\beta_i(t_k)} = 2A(0)\sum_{i_2,t_{k_2}}\rho_{\chi_{i_1},\chi_{i_2}}\sigma_{F_i}(t_k)\sigma_{F_{i_2}}(t_{k_2})min(t_k,t_{k_2})\beta_{i_2}(t_{k_2}) + 2A(0)\sum_{i_2,t_l}\rho_{F_i,\chi_{i_2}}\sigma_{Fi}(t_k)\sigma_{\chi_{i_2}}(t_l)min(t_k,t_l)\gamma_{i_2}(t_l) \tag{114}$$

and

$$\frac{\partial A(0)\gamma_i(t_l)}{\partial F_i} = \sum_{t_k} u_i(t_k,t_l)\chi_i^0(t_l) \tag{115}$$

and

$$\frac{\partial A(0)\beta_i(t_k)}{\partial F_i} = \sum_{tl} u_i(t_k,t_l)\chi_{i_1}^0(t_l) \tag{116}$$

and

$$T\frac{\partial A(0)^2\sigma^2}{\partial\sigma_{F_i}(t_k)} = 2A(0)^2\sum_{i_2,t_{k_2}}\rho_{F_i,F_{i_2}}\sigma_{F_{i_2}}(t_{k_2})min(t_k,t_{k_2})\beta_i(t_k)\beta_{i_2}(t_{k_2}) + 2A(0)^2\sum_{i_2,t_l}\rho_{F_i,\chi_{i_2}}\sigma_{\chi_{i_2}}(t_l)min(t_k,t_l)\gamma_{i_2}(t_l)\beta_i(t_k) \tag{117}$$

and

$$\frac{\partial\sigma_{F_j}(t_k)}{\partial F_i} = -\frac{K_{F_j}}{A(0)}\frac{\partial\sigma_{F_j}(t_k)}{\partial K_{F_j}}\frac{\partial A(0)}{\partial F_i} + \mathbb{1}_{i=j}\left(\frac{K}{A(0)}\frac{\partial\sigma_{F_i}(t_k)}{\partial K_{F_i}} + \frac{\partial\sigma_{F_i|K_{F_i}}(t_k)}{\partial F_i}\right) \tag{118}$$

and

$$\frac{\partial\sigma_{\chi_j(t_l)}}{\partial F_i} = -\mathbb{1}_{precious}\frac{K_{\chi_j}}{A(0)}\frac{\partial\sigma_{\chi_j}(t_l)}{\partial K_{\chi_j}}\frac{\partial U}{\partial F_i} \tag{119}$$

and $\frac{\partial\sigma_{F_i}(t_k)}{\partial K_{F_i}}$, $\frac{\partial\sigma_{F_i|K_{F_i}}(t_k)}{\partial F_i}$ and $\frac{\partial\sigma_{\chi_j}(t_l)}{\partial K_{\chi_j}}$ are computed by finite differences.

4.3 Total Delta Forex

Total deltas with respect to forex are given by :

$$\frac{\partial\sigma^2}{\partial\chi_i} = \frac{1}{A(0)^2}\left(\frac{\partial A(0)^2\sigma^2}{\partial\chi_i} - \sigma^2\frac{\partial A(0)^2}{\partial\chi_i}\right) \tag{120}$$

This decomposition will simplify calculation as $A(0)\beta_i$ and $A(0)\gamma_i$ are linear functions of χ_i.

$$\frac{\partial A(0)^2}{\partial\chi_i} = 2A(0)\sum_{j|C_j=C_i,t_k,t_l} u_j(t_k,t_l)F_j(t_k)\frac{\chi_i^0(t_l)}{\chi_i(t)} \tag{121}$$

and, as σ depends on χ_i trough β_i, γ_i, the correlation between forex, the volatility of each future and of each forex :

$$\begin{aligned}\frac{\partial A(0)^2\sigma^2}{\partial\chi_i} &= \sum_{j|C_j=C_i,t_l}\frac{\partial A(0)^2\sigma^2}{\partial A(0)\gamma_j(t_l)}\frac{\partial A(0)\gamma_j(t_l)}{\partial\chi_i} \\ &+ \sum_{j|C_j=C_i,t_k}\frac{\partial A(0)^2\sigma^2}{\partial A(0)\beta_j(t_k)}\frac{\partial A(0)\beta_j(t_k)}{\partial\chi_i} + \sum_j \frac{\partial A(0)^2\sigma^2}{\partial\rho_{\chi_i,\chi_j}}\frac{\partial\rho_{\chi_i,\chi_j}}{\partial\chi_i} \\ &+ \sum_{j,t_l}\frac{\partial A(0)^2\sigma^2}{\partial\sigma_{\chi_j}(t_l)}\frac{\partial\sigma_{\chi_j}(t_l)}{\partial\chi_i} + \sum_{j,t_k}\frac{\partial A(0)^2\sigma^2}{\partial\sigma_{F_j}(t_k)}\frac{\partial\sigma_{F_j}(t_k)}{\partial\chi_i}\end{aligned} \tag{122}$$

and

$$\frac{\partial A(0)\gamma_j(t_l)}{\partial\chi_i} = \mathbb{1}_{C_j=C_i}\sum_{t_k} u_j(t_k,t_l)F_j(t_k)\frac{\chi_i^0(t_l)}{\chi_i(0)} \tag{123}$$

and

$$\frac{\partial A(0)\beta_j(t_k)}{\partial\chi_i} = \mathbb{1}_{C_j=C_i}\sum_{t_l} u_j(t_k,t_l)F_j(t_k)\frac{\chi_i^0(t_l)}{\chi_i(0)} \tag{124}$$

and

$$\begin{aligned}T\frac{\partial A(0)^2\sigma^2}{\partial\sigma_{\chi_i}(t_l)} &= 2A(0)^2\sum_{j|C_j=C_i,i_2,t_{l_2}}\rho_{\chi_i,\chi_{i_2}}\sigma_{\chi_{i_2}}(t_{l_2})min(t_l,t_{l_2})\gamma_j(t_l)\gamma_{i_2}(t_{l_2}) \\ &+2A(0)^2\sum_{j|C_j=C_i,i_2,t_k}\rho_{\chi_i,F_{i_2}}\sigma_{F_{i_2}}(t_k)min(t_l,t_k)\gamma_j(t_l)\beta_{i_2}(t_k)\end{aligned} \tag{125}$$

and

$$T\frac{\partial A(0)^2\sigma^2}{\partial\rho_{\chi_i,\chi j}} = 2A(0)^2\sum_{\substack{i_1|C_{i_1}=C_i,\\ i_2|C_{i_2}=C_j,\\ t_{l_1},t_{l_2}}}\sigma_{\chi_{i_1}}(t_{l_1})\sigma_{\chi_{i_2}}(t_{l_2})min(t_{l_1},t_{l_2})\gamma_{i_1}(t_{l_1})\gamma_{i_2}(t_{l_2}) \tag{126}$$

and $\frac{\partial\rho_{\chi_i,\chi j}}{\partial\chi_i}$ is computed by finite differences.

and

$$\frac{\partial\sigma_{F_j}(t_k)}{\partial\chi i} = -\frac{K_{F_j}}{A(0)}\frac{\partial\sigma_{F_j}(t_k)}{\partial K_{F_j}}\frac{\partial A(0)}{\partial\chi_i} \tag{127}$$

and for j precious metal :

$$\frac{\partial\sigma_{\chi_j}(t_l)}{\partial\chi_i} = -\frac{K_{\chi_j}}{A(0)}\frac{\partial\sigma_{\chi_j}(t_l)}{\partial K_{\chi_j}}\frac{\partial A(0)}{\partial\chi_i} + \mathbb{1}_{i=j}\left(\frac{K}{A(0)}\frac{\partial\sigma_{\chi_i}(t_l)}{\partial K_{\chi_i}} + \frac{\partial\sigma_{\chi_i|K_{\chi_i}}(t_l)}{\partial\chi_i}\right) \tag{128}$$

and $\frac{\partial \sigma_{F_j}(t_k)}{\partial K_{F_j}}$, $\frac{\partial \sigma_{\chi_i}(t_l)}{\partial K_{\chi_i}}$ and $\frac{\partial \sigma_{\chi_i|K_{\chi_i}}(t_l)}{\partial \chi_i swp}$ are computed by finite differences.

If j does not correspond to a precious metal, $\frac{\partial \sigma_{\chi_j}(t_l)}{\partial \chi_i} = 0$ for $i \neq j$ and $\frac{\partial \sigma_{\chi_j}(t_l)}{\partial \chi_j}$ is computed by finite difference. The difference made between precious metal and other comes from the definition of the strike for volatility.

4.4 LME

Futures on commodities traded on the London Metal Exchange (LME) require a special treatment. Delta are computed for only some of the futures called Delta Futures and Vega are computed only for some of the futures called Vega futures. Delta are in fact computed for all futures and then dispatched on delta futures :

$$\Delta_{F_{j^\delta}} = \sum_i \Delta_{F_i} \frac{\partial F_i}{\partial F_{j^\delta}} \tag{129}$$

where

$$\frac{\partial F_i}{\partial F_{j^\delta}} = \mathbb{1}_{T_{j^\delta} < T_i < T_{j^\delta+1}} \frac{T_i - T_{j^\delta}}{T_{j^\delta+1} - T_{j^\delta}} + \mathbb{1}_{T_{j^\delta-1} < T_i < T_{j^\delta}} \frac{T_{j^\delta} - T_i}{T_{j^\delta} - T_{j^\delta-1}} + \mathbb{1}_{T_{j^\delta} = T_i} \tag{130}$$

and T stands for the delivery date.

γ and β coefficients are only computed for the vega future using the following formulas :

$$\gamma_{j_\sigma}(t_l) = \sum_{i, t_k \in T_F} \alpha_i(t_k, t_l) \frac{\partial \sigma_{j_\sigma}}{\partial \sigma_i} \tag{131}$$

$$\beta_{j_\sigma}(t_k) = \sum_{i, t_l \in T_\chi} \alpha_i(t_k, t_l) \frac{\partial \sigma_{j_\sigma}}{\partial \sigma_i} \tag{132}$$

where

$$\frac{\partial \sigma_{j_\sigma}}{\partial \sigma_i} = \mathbb{1}_{T_{j^\sigma} < T_i < T_{j^\sigma+1}} \frac{T_i - T_{j^\sigma}}{T_{j^\sigma+1} - T_{j^\sigma}} + \mathbb{1}_{T_{j^\sigma-1} < T_i < T_{j^\sigma}} \frac{T_{j^\sigma} - T_i}{T_{j^\sigma} - T_{j^\sigma-1}} + \mathbb{1}_{T_{j^\sigma} = T_i} \tag{133}$$

and T stands for the delivery date.

So in the LME case we finally obtain :

$$\begin{aligned} \frac{\partial A(0)^2\sigma^2}{\partial F_i} &= \sum_{j_\sigma, t_l} \frac{\partial A(0)^2\sigma^2}{\partial A(0)\gamma_{j_\sigma}(t_l)} \frac{\partial A(0)\gamma_{j_\sigma}(t_l)}{\partial F_i} + \sum_{j_\sigma} \frac{\partial A(0)^2\sigma^2}{\partial A(0)\beta_{j_\sigma}(t_k)} \frac{\partial A(0)\beta_{j_\sigma}(t_k)}{\partial F_i} \\ &+ \sum_j \frac{\partial A(0)^2\sigma^2}{\partial \sigma_{F_j}(t_k)} \frac{\partial \sigma_{F_j}(t_k)}{\partial F_i} + \sum_{j, t_l} \frac{\partial A(0)^2\sigma^2}{\partial \sigma_{\chi_j}(t_l)} \frac{\partial \sigma_{\chi_j}(t_l)}{\partial F_i} \end{aligned} \tag{134}$$

where

$$\frac{\partial A(0)\gamma_{j_\sigma}(t_l)}{\partial F_i} = \sum_{t_k} u_i(t_k, t_l)\chi_{i_1}(t_l)\frac{\partial \sigma_{j_\sigma}}{\partial \sigma_i} \tag{135}$$

and

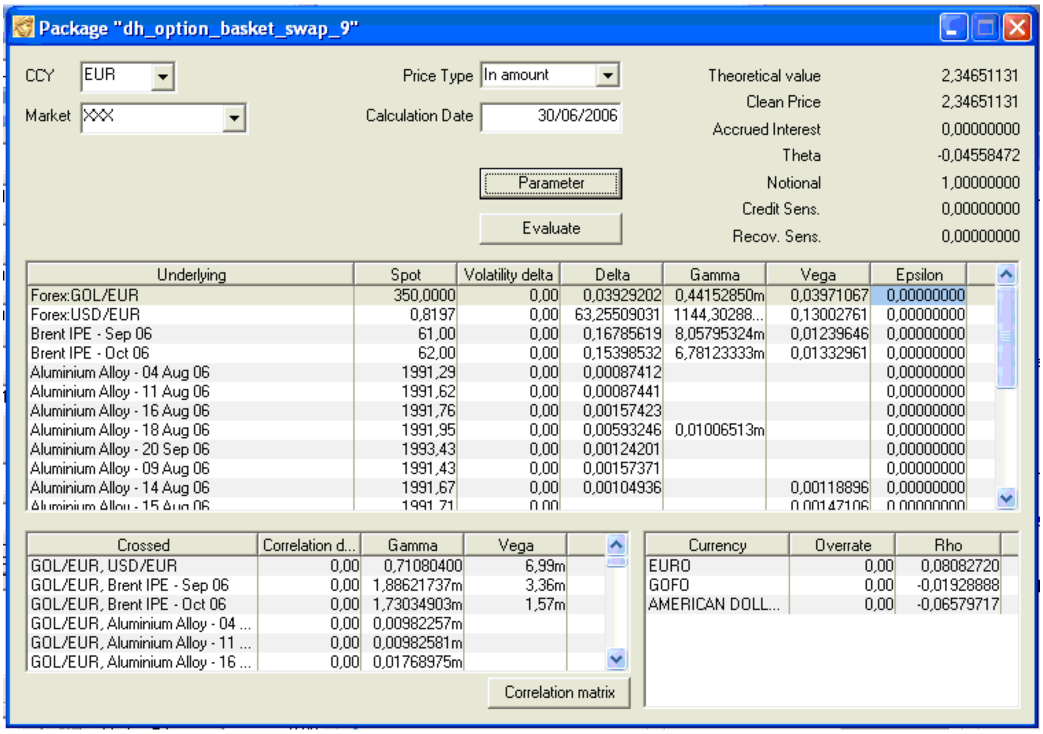
$$\frac{\partial A(0)\beta_{j_\sigma}(t_k)}{\partial F_i} = \sum_{tl \in T_\chi} u_i(t_k, t_l)\chi_{i_1}(t_l)\frac{\partial \sigma_{j_\sigma}}{\partial \sigma_i} \tag{136}$$

4.5 Validation

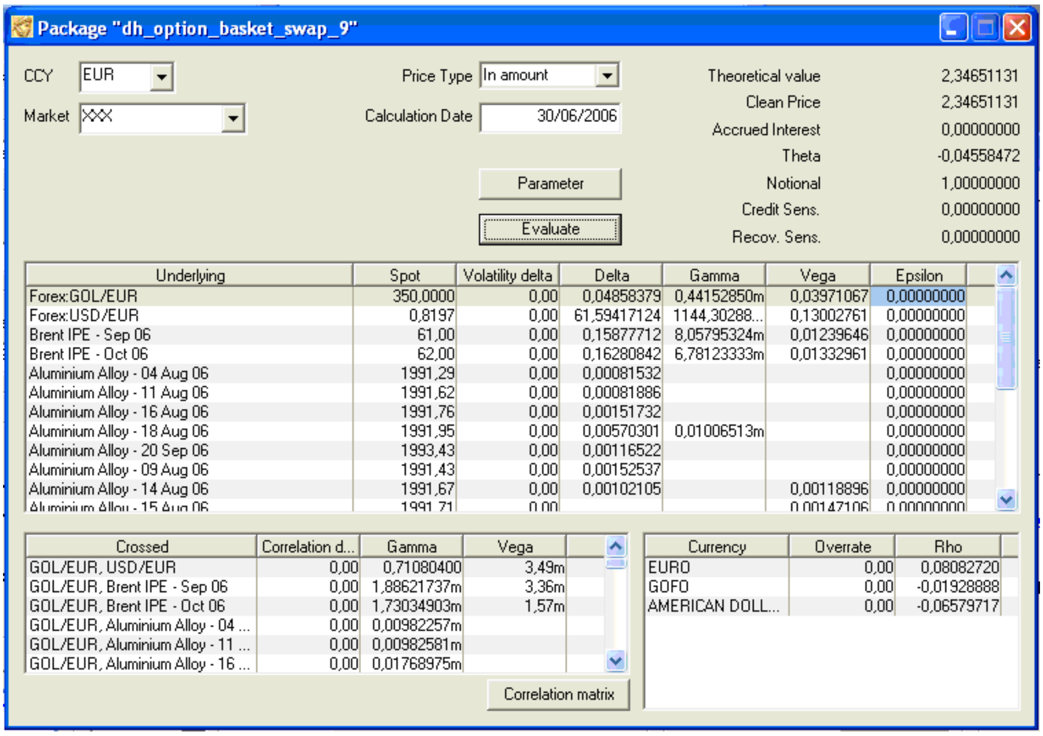
The optimized computation of delta total has been implemented in Sophis software. To validate the results, we configured an option on a basket containing Brent (quoted in Dollar), Gold (Precious Metal) and Aluminium (LME quoted in Dollar). The currency of the option and of its strike is set to Euro.

Then we computed partial deltas, total deltas by optimized computation and total delta by finite differences. Here are the results of these computations :

- partial deltas



- total deltas by optimized computation



Underlying	Spot	Volatility delta	Delta	Gamma	Vega	Epsilon
Forex:GOL/EUR	350,0000	0,00	0,04858379	0,44152850m	0,03971067	0,00000000
Forex:USD/EUR	0,8197	0,00	61,59417124	1144,30288...	0,13002761	0,00000000
Brent IPE - Sep 06	61,00	0,00	0,15877712	8,05795324m	0,01239646	0,00000000
Brent IPE - Oct 06	62,00	0,00	0,16280842	6,78123333m	0,01332961	0,00000000
Aluminium Alloy - 04 Aug 06	1991,29	0,00	0,00081532			0,00000000
Aluminium Alloy - 11 Aug 06	1991,62	0,00	0,00081886			0,00000000
Aluminium Alloy - 16 Aug 06	1991,76	0,00	0,00151732			0,00000000
Aluminium Alloy - 18 Aug 06	1991,95	0,00	0,00570301	0,01006513m		0,00000000
Aluminium Alloy - 20 Sep 06	1993,43	0,00	0,00116522			0,00000000
Aluminium Alloy - 09 Aug 06	1991,43	0,00	0,00152537			0,00000000
Aluminium Alloy - 14 Aug 06	1991,67	0,00	0,00102105		0,00118896	0,00000000
Aluminium Alloy - 15 Aug 06	1991,71	0,00			0,00147106	0,00000000

Crossed	Correlation d...	Gamma	Vega
GOL/EUR, USD/EUR	0,00	0,71080400	3,49m
GOL/EUR, Brent IPE - Sep 06	0,00	1,88621737m	3,36m
GOL/EUR, Brent IPE - Oct 06	0,00	1,73034903m	1,57m
GOL/EUR, Aluminium Alloy - 04 ...	0,00	0,00982257m	
GOL/EUR, Aluminium Alloy - 11 ...	0,00	0,00982581m	
GOL/EUR, Aluminium Alloy - 16 ...	0,00	0,01768975m	

Currency	Overrate	Rho
EURO	0,00	0,08082720
GOFO	0,00	-0,01928888
AMERICAN DOLL...	0,00	-0,06579717

- total deltas by finite differences :

Risk sources	Price +	Price -	Bump	Delta by finite differences	Delta Total optimized
GOL/EUR	2,35137319	2,34165516	0.1	0,04859015	0.04858379
USD/EUR	2,35439556	2,34207216	0.0001	61,617	61.59417124
Brent Sep 2006	2,34809948	2,34492394	0.01	0,1587770	0.15877712
Brent Oct 2006	2,34813981	2,34488364	0.01	0,1628085	0.16280842
AA 04 Aug 2006	2,34659634	2,34643328	0.1	0,00081530	0.00081532
AA 11 Aug 2006	2,34659398	2,34643021	0.1	0,00081885	0.00081886
AA 16 Aug 2006	2,34666016	2,34635669	0.1	0,00151735	0.00151732
AA 18 Aug 2006	2,34706808	2,34592749	0.1	0,00570295	0.00570301
AA 20 Sep 2006	2,34662950	2,34639646	0.1	0,00116520	0.00116522
AA 09 Aug 2006	2,34666603	2,34636096	0.1	0,00152535	0.00152537
AA 14 Aug 2006	2,34661682	2,34641261	0.1	0,00102105	0.00102105
AA 31 Jul 2006	2,34654515	2,34647748	0.1	0,00033835	0.00033835
AA 07 Aug 2006	2,34660754	2,34640846	0.1	0,00099540	0.00099540
AA 02 Aug 2006	2,34666904	2,34636793	0.1	0,00150555	0.00150556

Comparison between Partial Deltas and Total Deltas shows that they are significantly different. So the correction term added for Optimized Total Deltas is validated by the comparison between Optimized Total Deltas and finite differences Total Deltas. As Optimized Total Deltas and finite differences Total Deltas are quite similar, the implementation of Optimized Total Deltas is validated.

5 ANNEXES

This section contains all the data used to generates the graphics presented in section experimentation.

Strike.

Absolute price.

Strike	Monte-Carlo	MC Moins	MC Plus	Sophis	Gentle	Levy	Ju	Curran
40	140,0004562	139,9967401	140,0041723	140,0000019	140,0000001	140,0000026	139,9999961	140
80	100,0223121	100,0186595	100,0259647	100,043371	100,0223451	100,0487614	100,0201111	100,0226717
120	61,68070643	61,67732603	61,68408683	61,89149981	61,53948673	61,98264522	61,68831474	61,67783417
160	31,93556142	31,93243331	31,93868953	32,08405853	31,26228726	32,31858601	31,93246406	31,93474978
200	14,54539171	14,54220224	14,54858118	14,3496091	13,5147618	14,61571641	14,52870103	14,54369155
240	6,1782671	6,17531463	6,18121957	5,81440646	5,25312374	6,01346804	6,17865867	6,17983723
280	2,55513409	2,55285039	2,55741779	2,2226075	1,9208285	2,34077301	2,56244177	2,55388153
320	1,05227034	1,05097532	1,05356536	0,82424045	0,68127224	0,88554832	1,05973176	1,05224059
360	0,438165	0,43744094	0,43888905	0,30198763	0,23900368	0,33134627	0,4412808	0,43831696
400	0,18580523	0,18540997	0,18620048	0,11059822	0,08396266	0,12400485	0,18566545	0,18600264

Relative delta to the reference price.

Strike	Monte-Carlo	MC Moins	MC Plus	Sophis	Gentle	Levy	Ju	Curran
40	0	-2,65436E-05	2,65436E-05	-3,24506E-06	-3,25792E-06	-3,23978E-06	-3,28635E-06	-3,25856E-06
80	0	-3,6518E-05	3,65179E-05	0,000210542	3,29726E-07	0,000264433	-2,20055E-05	3,5947E-06
120	0	-5,48048E-05	5,48048E-05	0,003417493	-0,002289528	0,00489519	0,00012335	-4,65666E-05
160	0	-9,79507E-05	9,79507E-05	0,004649898	-0,021082271	0,01199367	-9,69878E-05	-2,54149E-05
200	0	-0,000219277	0,000219277	-0,013460113	-0,070856112	0,004834844	-0,001147489	-0,000116887
240	0	-0,00047788	0,00047788	-0,05889364	-0,149741561	-0,026673994	6,33786E-05	0,000254138
280	0	-0,000893769	0,000893769	-0,130140563	-0,248247477	-0,083894259	0,002859999	-0,000490213
320	0	-0,001230691	0,001230691	-0,216702763	-0,352569189	-0,158440292	0,007090782	-2,82722E-05
360	0	-0,001652483	0,00165246	-0,310790159	-0,454534981	-0,243786542	0,00711102	0,00034681
400	0	-0,002127281	0,002127228	-0,404762611	-0,548114657	-0,332608399	-0,000752293	0,001062457

Volatility

Absolute price

Volatility	Monte-Carlo	MC Moins	MC Plus	Sophis	Gentle	Levy	Ju	Curran
0,1	5,46709403	5,466025	5,46816306	5,46697607	5,45277703	5,47113338	5,46714807	5,46741666
0,2	10,92949566	10,92707098	10,93192033	10,9260143	10,81294589	10,95931774	10,92736795	10,92933643
0,3	16,37617486	16,37145201	16,38089771	16,36923935	15,99055713	16,48188739	16,37369466	16,37985151
0,4	21,81137266	21,80314065	21,81960466	21,78884152	20,900814	22,05666083	21,79908562	21,81216254
0,5	27,22047404	27,20725042	27,23369766	27,17711115	25,4663936	27,70211875	27,19651394	27,21860386
0,6	32,59396878	32,5738948	32,61404275	32,52647208	29,61947432	33,43747964	32,55919425	32,59072976
0,7	37,87737017	37,84837104	37,9063693	37,82951403	33,30339506	39,2825996	37,88094213	37,91950124
0,8	43,22750517	43,18627759	43,26873276	43,07902367	36,47389075	45,25758597	43,15662025	43,19551164
0,9	48,42731953	48,37057512	48,48406394	48,26801426	39,09986876	51,38196942	48,38255569	48,40922387
1	53,55129694	53,47515593	53,62743795	53,38975356	41,1637131	57,67323911	53,5567262	53,55120238

Relative Delta to the reference price.

Volatility	Monte-Carlo	MC Moins	MC Plus	Sophis	Gentle	Levy	Ju	Curran
0,1	0	-0,000195539	0,000195539	-2,15764E-05	-0,002618759	0,000738848	9,88459E-06	5,90131E-05
0,2	0	-0,000221847	0,000221846	-0,000318529	-0,010663783	0,002728587	-0,000194676	-1,45688E-05
0,3	0	-0,000288398	0,000288398	-0,000423512	-0,023547485	0,006455264	-0,000151452	0,000224512
0,4	0	-0,000377418	0,000377418	-0,001033	-0,041746967	0,011245884	-0,000563332	3,62141E-05
0,5	0	-0,000485797	0,000485797	-0,001593025	-0,064439746	0,017694207	-0,000880223	-6,87049E-05
0,6	0	-0,00061588	0,00061588	-0,002070834	-0,091259045	0,025879354	-0,001066901	-9,93748E-05
0,7	0	-0,000765606	0,000765606	-0,001263449	-0,120757463	0,037099445	9,43033E-05	0,001112302
0,8	0	-0,000953735	0,000953735	-0,003434885	-0,156234194	0,04696271	-0,001639811	-0,00074012
0,9	0	-0,001171744	0,001171744	-0,003289574	-0,192607207	0,061012047	-0,000924351	-0,000373666
1	0	-0,001421833	0,001421833	-0,00301661	-0,231321827	0,076971846	0,000101384	-1,76578E-06

Relative spot.

Absolute price.

Offset	Monte-Carlo	MC Moins	MC Plus	Sophis	Gentle	Levy	Ju	Curran
0	21,8890952	21,88193747	21,89625294	21,85671431	20,96554617	22,12372465	21,86923595	21,88156578
10	21,92745492	21,91921393	21,9356959	21,90099185	21,01197893	22,16895606	21,90948915	21,92251095
20	22,04442857	22,03331724	22,0555399	22,03327151	21,15084775	22,3039742	22,03032371	22,04524146
30	22,24734959	22,2320812	22,26261799	22,25192732	21,38089324	22,52679246	22,23187972	22,24942234
40	22,52418666	22,50363409	22,54473924	22,55435476	21,70010881	22,83423267	22,5141375	22,53442842
50	22,9021307	22,87517729	22,9290841	22,93711407	22,10586621	23,22210516	22,87662757	22,89924057
60	23,35217927	23,31818867	23,38616988	23,39610507	22,59506831	23,68542701	23,31816555	23,342297
70	23,83403997	23,79345096	23,87462899	23,9267545	23,16431216	24,21865423	23,83668423	23,86131285
80	24,40640309	24,36162297	24,45118322	24,52419852	23,8100473	24,81590623	24,42919751	24,45312914
90	25,08208165	25,03559734	25,12856595	25,18344759	24,52871784	25,47116655	25,09189175	25,11370933

Relative delta to the reference price.

Offset	Monte-Carlo	MC Moins	MC Plus	Sophis	Gentle	Levy	Ju	Curran
0	0	-0,000327	0,000327	-0,001479316	-0,042192198	0,010719011	-0,000907267	-0,00034398
10	0	-0,00037583	0,000375829	-0,001206846	-0,041750216	0,011013642	-0,000819328	-0,000225469
20	0	-0,000504043	0,000504043	-0,000506117	-0,040535449	0,011773752	-0,000639838	3,68751E-05
30	0	-0,000686302	0,000686302	0,000205765	-0,038946498	0,012560726	-0,000695358	9,31684E-05
40	0	-0,000912467	0,000912467	0,001339365	-0,036586353	0,013765026	-0,00044615	0,000454701
50	0	-0,001176895	0,001176895	0,001527516	-0,03476814	0,013971384	-0,00111357	-0,000126195
60	0	-0,001455564	0,001455565	0,001881015	-0,032421426	0,01427052	-0,001456554	-0,000423184
70	0	-0,001702985	0,001702985	0,003890005	-0,028099634	0,016137183	0,000110945	0,001144283
80	0	-0,001834769	0,00183477	0,004826415	-0,024434399	0,016778513	0,000933952	0,0019145
90	0	-0,001853288	0,001853287	0,004041369	-0,022062117	0,015512464	0,00039112	0,001260967

Maturity.

Absolute price.

Maturity	Monte-Carlo	MC Moins	MC Plus	Sophis	Gentle	Levy	Ju	Curran
0,2	6,91499881	6,91364944	6,91634818	6,91423271	6,88552245	6,92264879	6,91457965	6,91509103
0,4	9,77552515	9,77345544	9,77759487	9,77441521	9,6934103	9,79823666	9,77538795	9,77681134
0,6	11,97015504	11,96737407	11,97293601	11,9665345	11,81808516	12,01032867	11,96830646	11,97086425
0,8	13,81678476	13,81327707	13,82029244	13,81242298	13,58443229	13,87989649	13,81512884	13,81897248
1	15,44363634	15,43938191	15,44789077	15,43679274	15,11894904	15,53115638	15,44054449	15,44578506
1,2	16,91473635	16,90972543	16,91974726	16,90362984	16,48683949	17,02776086	16,90852416	16,91524376
1,4	18,26498186	18,25919163	18,27077209	18,25096022	17,7270333	18,4074917	18,25708266	18,26534141
1,6	19,51441661	19,50784974	19,52098348	19,50356981	18,8650238	19,6949461	19,51099744	19,52083803
1,8	20,70403039	20,69662304	20,71143774	20,67868187	19,91860898	20,90719579	20,68748514	20,6989358
2	21,82104333	21,81278945	21,82929721	21,78884152	20,900814	22,05666083	21,79908562	21,81216254

Relative delta to the reference price.

Maturity	Monte-Carlo	MC Moins	MC Plus	Sophis	Gentle	Levy	Ju	Curran
0,2	0	-0,000195137	0,000195137	-0,000110788	-0,00426267	0,001106288	-6,06161E-05	1,33362E-05
0,4	0	-0,000211724	0,000211725	-0,000113543	-0,008400045	0,002323303	-1,40351E-05	0,000131572
0,6	0	-0,000232325	0,000232325	-0,000302464	-0,012704086	0,00335615	-0,000154432	5,92482E-05
0,8	0	-0,000253872	0,000253871	-0,000315687	-0,016816682	0,004567758	-0,000119848	0,000158338
1	0	-0,000275481	0,000275481	-0,000443134	-0,021024019	0,005667062	-0,000200202	0,000139133
1,2	0	-0,000296246	0,000296245	-0,000656617	-0,025297282	0,006682014	-0,000367265	2,99981E-05
1,4	0	-0,000317013	0,000317013	-0,000767679	-0,029452455	0,007802353	-0,000432478	1,96852E-05
1,6	0	-0,000336514	0,000336514	-0,000555835	-0,033277593	0,009251083	-0,000175213	0,00032906
1,8	0	-0,000357773	0,000357773	-0,001224328	-0,037935677	0,009812843	-0,000799132	-0,000246068
2	0	-0,000378253	0,000378253	-0,001475723	-0,042171647	0,010797719	-0,001006263	-0,000406983

Maturity.

Absolute price.

Maturity	Monte-Carlo	MC Moins	MC Plus	Sophis	Gentle	Levy	Ju	Curran
2	21,82006716	21,81182358	21,82831073	21,78884152	20,900814	22,05666083	21,79908562	21,81216254
4	30,76789734	30,75040281	30,78539188	30,6957116	28,24462586	31,45805771	30,72333159	30,75172749
6	37,54473695	37,5162471	37,57322679	37,45047186	33,05555663	38,85885025	37,50033931	37,5385574
8	43,20406412	43,1629113	43,24521694	43,07902367	36,47389075	45,25758597	43,15662025	43,19551164
10	48,10831129	48,05222858	48,16439399	47,98055007	38,96821599	51,03642574	48,09267475	48,12047475
12	52,47966671	52,40737083	52,5519626	52,36068918	40,79244302	56,3879701	52,51555085	52,51854097
14	56,57988142	56,48715304	56,67260979	56,34214568	42,10530366	61,42358388	56,54910135	56,51225978
16	60,13225933	60,01826483	60,24625382	60,00522907	43,01516368	66,212892	60,27428149	60,18200373
18	63,42858868	63,29578999	63,56138736	63,40597673	43,60052704	70,80138669	63,74711065	63,58400154
20	66,70245299	66,53957537	66,8653306	66,58537697	43,92072143	75,21941201	67,00780487	66,75950938

Relative Delta to the reference price.

Maturity	Monte-Carlo	MC Moins	MC Plus	Sophis	Gentle	Levy	Ju	Curran
2	0	-0,000377798	0,000377798	-0,001431052	-0,042128796	0,01084294	-0,000961571	-0,000362264
4	0	-0,000568597	0,000568597	-0,002346138	-0,082009877	0,022431184	-0,00144845	-0,000525543
6	0	-0,000758824	0,000758824	-0,00251074	-0,119568831	0,035001265	-0,001182526	-0,000164592
8	0	-0,000952522	0,000952522	-0,002894183	-0,155776395	0,047530756	-0,001098134	-0,000197955
10	0	-0,001165759	0,001165759	-0,0026557	-0,189989943	0,060865043	-0,000325028	0,000252835
12	0	-0,001377598	0,001377598	-0,002267117	-0,222700036	0,074472717	0,000683772	0,000740749
14	0	-0,001638893	0,001638893	-0,004201772	-0,255825523	0,085608212	-0,000544011	-0,001195153
16	0	-0,00189573	0,001895729	-0,002112514	-0,284657451	0,101120975	0,00236183	0,00082725
18	0	-0,002093672	0,002093672	-0,000356495	-0,31260449	0,116237775	0,005021741	0,002450202
20	0	-0,002441854	0,002441853	-0,001755198	-0,341542635	0,127685844	0,004577821	0,000855387

Relative volatility.

Absolute price.

Volatility x	Monte-Carlo	MC Moins	MC Plus	Sophis	Gentle	Levy	Ju	Curran
0,04	11,86689218	11,86320131	11,87058305	11,95021347	11,64676145	12,32603369	11,69886845	11,86442168
0,08	12,73279638	12,72918834	12,73640442	12,81675764	12,49910709	13,1505464	12,61267214	12,72973379
0,12	13,69538783	13,69174564	13,69903003	13,77003218	13,4291204	14,06425501	13,61377391	13,69178351
0,16	14,72632685	14,72257198	14,73008171	14,79295481	14,41849179	15,05199329	14,68108288	14,73152708
0,2	15,83210054	15,82800497	15,83619611	15,87171616	15,45188898	16,10204845	15,79890196	15,8313499
0,24	16,97240442	16,9678389	16,97696994	16,99531809	16,51658554	17,20567222	16,95522525	16,97649147
0,28	18,15570591	18,15049148	18,16092034	18,15505224	17,60200964	18,35655126	18,14054903	18,15525383
0,32	19,35946806	19,35343005	19,36550608	19,34402262	18,69932013	19,5503297	19,34711504	19,35855136
0,36	20,57541796	20,56838895	20,58244696	20,55674735	19,80104876	20,784214	20,56844903	20,57934826
0,4	21,81214768	21,80391878	21,82037657	21,78884152	20,900814	22,05666083	21,79908562	21,81216254

Relative delta to the reference price.

Volatility x	Monte-Carlo	MC Moins	MC Plus	Sophis	Gentle	Levy	Ju	Curran
0,04	0	-0,000311022	0,000311022	0,007021324	-0,01854999	0,038690965	-0,014159034	-0,000208184
0,08	0	-0,000283366	0,000283366	0,006594094	-0,018353336	0,032808977	-0,009434239	-0,000240528
0,12	0	-0,000265943	0,000265944	0,005450328	-0,019442124	0,026933679	-0,005959227	-0,000263178
0,16	0	-0,000254977	0,000254976	0,004524411	-0,020903723	0,022114574	-0,003072319	0,000353125
0,2	0	-0,000258688	0,000258688	0,002502234	-0,024015232	0,017050669	-0,002096916	-4,74125E-05
0,24	0	-0,000268997	0,000268997	0,001350054	-0,026856471	0,013743945	-0,001012182	0,000240806
0,28	0	-0,000287206	0,000287206	-3,60036E-05	-0,030497094	0,011062382	-0,000834827	-2,49002E-05
0,32	0	-0,000311889	0,00031189	-0,000797824	-0,034099487	0,009858827	-0,000638087	-4,73515E-05
0,36	0	-0,000341622	0,000341621	-0,000907423	-0,037635649	0,01014784	-0,000338702	0,000191019
0,4	0	-0,000377262	0,000377262	-0,001068495	-0,041781015	0,011209953	-0,000598843	6,81272E-07

Correlation.

Absolute price.

Correlation	Monte-Carlo	MC Moins	MC Plus	Sophis	Gentle	Levy	Ju	Curran
-0,8	10,0279798	10,01178556	10,04417405	8,39692608	7,74238779	10,69284391	10,113593	9,87735059
-0,6	12,59541685	12,58022121	12,61061248	11,53461513	10,70039783	12,97667951	12,63870448	12,5062582
-0,4	14,68558013	14,6715677	14,69959256	13,98068502	13,04870743	14,95272264	14,71290031	14,63976559
-0,2	16,51103636	16,49831747	16,52375524	16,05444792	15,07567076	16,7310018	16,52919909	16,49287658
0	18,17686851	18,1654282	18,18830881	17,88588461	16,89794605	18,3695964	18,17341518	18,16036884
0,2	19,69030008	19,68019967	19,70040048	19,54324807	18,576439	19,90354533	19,69274321	19,69419314
0,4	21,12763341	21,11879338	21,13647344	21,0676673	20,14765004	21,35588172	21,11655478	21,12672456
0,6	22,48165119	22,47398707	22,48931532	22,48620293	21,63546825	22,74268554	22,46466626	22,48000971
0,8	23,76736218	23,76071116	23,7740132	23,81778675	23,05652232	24,07569478	23,75122912	23,77004363
1	25,00837984	25,00237878	25,0143809	25,07627973	24,42292813	25,36377794	24,98678443	25,00865665

Relative delta to the reference price.

Correlation	Monte-Carlo	MC Moins	MC Plus	Sophis	Gentle	Levy	Ju	Curran
-0,8	0	-0,001614906	0,001614907	-0,16265028	-0,227921481	0,066300902	0,008537432	-0,015020893
-0,6	0	-0,001206442	0,001206441	-0,084221248	-0,150453061	0,030269952	0,003436776	-0,007078658
-0,4	0	-0,000954163	0,000954163	-0,047999133	-0,111461222	0,018190804	0,001860341	-0,003119696
-0,2	0	-0,000770327	0,000770326	-0,02765353	-0,086933707	0,013322328	0,001100036	-0,001099857
0	0	-0,000629388	0,000629388	-0,016008473	-0,07035989	0,01060292	-0,000189985	-0,000907729
0,2	0	-0,000512964	0,000512963	-0,007468246	-0,056569025	0,010829964	0,000124078	0,000197715
0,4	0	-0,000418411	0,000418411	-0,002838279	-0,046383963	0,010803307	-0,000524367	-4,30171E-05
0,6	0	-0,000340906	0,000340906	0,000202465	-0,037638825	0,011610995	-0,000755502	-7,30142E-05
0,8	0	-0,000279838	0,000279838	0,002121589	-0,029908235	0,012972942	-0,000678791	0,000112821
1	0	-0,000239962	0,000239962	0,002715086	-0,023410221	0,014211161	-0,000863527	1,10687E-05

Number of fixing date.

Absolute price

n	Monte-Carlo	MC Moins	MC Plus	Sophis	Gentle	Levy	Ju	Curran
10	21,81115526	21,80624776	21,81606276	21,78884152	20,900814	22,05666083	21,79908562	21,81216254
40	20,67975296	20,67508806	20,68441786	20,66200683	19,84555253	20,93413658	20,65888754	20,67721149
70	20,51765484	20,51302125	20,52228843	20,500401	19,69433375	20,77323402	20,49545421	20,51462463
100	20,44961506	20,44499747	20,45423265	20,43571227	19,63380957	20,70883292	20,43003947	20,44955592
130	20,41676253	20,4121512	20,42137386	20,40086878	19,60121076	20,67414581	20,39480619	20,41451076
160	20,3944283	20,38981975	20,39903684	20,37908764	19,58083332	20,65246294	20,37278182	20,39260461
190	20,38158479	20,37697111	20,38619847	20,36418299	19,56688945	20,63762577	20,35771093	20,37761489
220	20,37024095	20,36563083	20,37485107	20,35334235	19,55674772	20,62683432	20,34674947	20,36671262
250	20,35826801	20,35367054	20,36286548	20,34510295	19,54903959	20,61863238	20,3384183	20,35842651
280	20,35132606	20,34672868	20,35592345	20,33862883	19,54298296	20,61218773	20,33187211	20,35191577

Relative delta to the reference price.

n	Monte-Carlo	MC Moins	MC Plus	Sophis	Gentle	Levy	Ju	Curran
10	0	-0,000225	0,000225	-0,001023043	-0,041737416	0,011255964	-0,00055337	4,61819E-05
40	0	-0,000225578	0,000225578	-0,00085814	-0,040338994	0,012301096	-0,001008978	-0,000122897
70	0	-0,000225834	0,000225834	-0,000840927	-0,040127446	0,012456549	-0,001082026	-0,000147688
100	0	-0,000225803	0,000225803	-0,000679856	-0,03989344	0,012675929	-0,00095726	-2,89199E-06
130	0	-0,00022586	0,00022586	-0,000778466	-0,039945205	0,012606469	-0,001075408	-0,00011029
160	0	-0,000225971	0,000225971	-0,000752199	-0,039893003	0,012652212	-0,001061392	-8,9421E-05
190	0	-0,000226365	0,000226365	-0,0008538	-0,039972129	0,012562369	-0,001171345	-0,000194779
220	0	-0,000226316	0,000226316	-0,000829573	-0,039935376	0,012596482	-0,001153225	-0,00017321
250	0	-0,000225828	0,000225828	-0,000646669	-0,039749375	0,012789122	-0,00097502	7,78553E-06
280	0	-0,000225901	0,000225901	-0,000623902	-0,039719431	0,01281792	-0,000955906	2,89765E-05

6 Bibliography

References

[Bla76] F. Black. The pricing of commodity contracts. *Journal of Financial Economics*, (3):167–179, March 1976.

[Cur94] M. Curran. Valuing asian and portefolio options by conditioning on the geometric mean price. *Management of Science*, 40(12):1705–1711, 1994.

[DLV04] "G. Deelstra, J. Liinev, and M. Vanmael". Pricing of arithmetic basket options by conditioning. *Insurance : Mathematics and Economics*, 34(1):55–77, 2004.

[FPTV92] B. Flannery, H. Press, S. Teukolsky, and W. Vetterling. *Numerical Recipies in C.* Cambridge University Press, second edition edition, 1992.

[Gen93] D. Gentle. Basket weaving. *RISK*, 6(6):59–61, June 1993.

[Ju02] N. Ju. Pricing asian and basket options via taylor expansion. *Journal of Computational Finance*, 5(3):79–103, June 2002.

[Lev92] E. Levy. Pricing average rate currency options. *Journal of International Money and Finance*, 11:474–491, 1992.

[RS95] L.C.G. Rogers and Z. Shi. The value of an asian option. *Journal of Applied Probability*, (32):1077–1088, 1995.

[Sac06] Goldman Sachs. The gsci manual. `http://www.gs.com/gsci/docs/GSCI_Manual_2006_FINAL.pdf`, 2006.

[Sop06] Sophis. Sophis financial models reference manual, June 2006.

[TW91] S.M. Turnbull and L.M. Wakeman. Quick algorithm for pricing europe average options. *Journal of Financial and Quantitive Analysis*, 26(3):377–389, 1991.